QCD and jets
Lecture 1: Foundations

Sebastian Sapeta

CERN

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The theory of QCD

The QCD Lagrangian

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{classical}} + \mathcal{L}_{\text{gauge-fixing}} + \mathcal{L}_{\text{ghost}}\,,$$

where

$$\mathcal{L}_{\text{classical}} = \sum_{\text{flavours}} \bar{q}_i \left(i\not{D} - m\right)_{ij} q_j - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}\,,$$

$i, j = 1, \ldots, N$ and $a = 1, \ldots, N^2-1$, where N is the number of colors.

Covariant derivative

$$\not{D} = \gamma_\mu D^\mu = \gamma_\mu \left(\partial^\mu + i g_s A^\mu\right) \qquad \text{with} \qquad A^\mu = A^\mu_a T^a\,,$$

where T^as are generators of the SU(N) color algebra

$$[T^a, T^b] = i f^{abc} T^c\,.$$

Field-strength tensor

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_s f^{abc} A^b_\mu A^c_\nu\,.$$

SU(N) color group

SU(N) is a special unitary Lie group defined by

$$[T^a, T^b] = if^{abc}T^c$$

- T^as are are the generators of SU(N) color algebra
- f^{abc} are the structure constants of SU(N)
  $\hookrightarrow f^{abc} \neq 0 \Rightarrow$ QCD is a non-abelian theory
- The group elements can be represented by $N \times N$ matrices
  $U = \exp(i\theta_a T^a)$ which are unitary $UU^\dagger = 1$ and with $\det(U) = 1$
  $\hookrightarrow$ representations of T^a operators are hermitian and traceless matrices

For QCD, $N = 3$

- quarks are in the fundamental, triplet representation

$$\left(T^a_{ij}\right)_F = \frac{1}{2}\lambda^a_{ij} \qquad \text{where } \lambda^a \text{ are the Gell-Mann matrices}$$

- gluons are in the adjoint, octet representation

$$(T^a_{bc})_A = -if^{abc}$$

SU(N) color group

Useful relations:

$$
\begin{aligned}
\mathrm{Tr}\,(T^a)_F\,(T^b)_F &= T_R\delta^{ab}\,, \qquad T_R = \frac{1}{2}\ \text{(by convention)} \\
\sum_a (T^a_{ij})_F\,(T^a_{jk})_F &= C_F\,\delta_{ik}\,, \qquad C_F = \frac{N^2-1}{2N}\,, \\
\mathrm{Tr}\left[(T^a)_A(T^b)_A\right] &= C_A\delta^{ab}\,, \qquad C_A = N\,,
\end{aligned}
$$

where C_F and C_A are the Casimirs of the fundamental and the adjoint representation, respectively ($T^a T^a$ is an invariant of the algebra).
For QCD we have

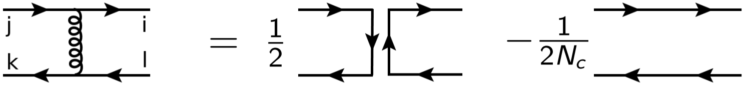
$$
C_F = \frac{4}{3}\,, \qquad C_A = 3
$$

And also

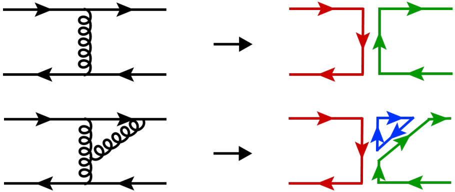
$$
(T^a_{ij})_F\,(T^a_{kl})_F = \frac{1}{2}\left[\delta_{jk}\delta_{il} - \frac{1}{2N}\delta_{ij}\delta_{kl}\right] \quad \text{(Fierz identity)}
$$

Fierz identity and the large N_c limit

$$\left(T^a_{ij}\right)_F \left(T^a_{kl}\right)_F = \frac{1}{2}\left[\delta_{jk}\delta_{il} - \frac{1}{2N_c}\delta_{ij}\delta_{kl}\right]$$



If $N_c \gg 1$, we can replace a gluon with the $q\bar{q}$ pair:



QCD Lagrangian: local gauge symmetry

$$\mathcal{L}_{\text{classical}} = \sum_{\text{flavours}} \bar{q}_i \left(i\not{D} - m\right)_{ij} q_j - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$

The above Lagrangian is invariant under the local gauge symmetry.

Redefinition of the quark fields by the SU(3) group element

$$U(x) = \exp\left(i\theta_a(x) T^a\right) ,$$

independently at each phase space point, does not change the physical content of the theory.

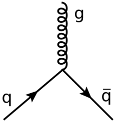
SU(3) transformation:

$$\begin{aligned}
q_i(x) &\mapsto q'_i(x) = U(x)_{ij} q_j(x) \\
D_\mu q(x) &\mapsto D'_\mu(x) = U(x)_{ij} D_\mu q_j(x) \\
A^\mu &\mapsto U(x) A^\mu U(x)^{-1} + \frac{i}{g_s}\left[\partial^\mu U(x)\right] U(x)^{-1} \\
F_{\mu\nu} &\mapsto U(x) F_{\mu\nu} U(x)^{-1}
\end{aligned}$$

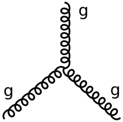
QCD Lagrangian: the interactions

$$\mathcal{L}_{\text{classical}} = \sum_{\text{flavours}} \bar{q}_i \Big(i\gamma_\mu \left(\partial^\mu + i g_s A^\mu\right) - m \Big)_{ij} q_j$$

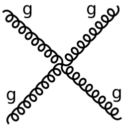
$$-\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_s f^{abc} A^b_\mu A^c_\nu\right)\left(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a - g_s f^{ade} A^\mu_d A^\nu_e\right)$$



$$-g_s\, \bar{q}_i \gamma_\mu A^\mu_{ij} q_j$$



$$\frac{g_s}{4} f^{abc} A^b_\mu A^c_\nu \left(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a\right)$$



$$-\frac{g_s g_s}{4} f^{abc} f^{ade} A^b_\mu A^c_\nu A^{d\,\mu} A^{e\,\nu}$$

Gauge-fixing and ghost terms

The QCD Lagrangian

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{classical}} + \mathcal{L}_{\text{gauge-fixing}} + \mathcal{L}_{\text{ghost}}$$

The gauge-fixing and ghost part:

$$\begin{aligned}
\mathcal{L}_{\text{gauge-fixing}} &= -\frac{1}{2\xi}\left(\partial_\mu A^{a\mu}\right)\left(\partial_\nu A^{a\nu}\right) \\
\mathcal{L}_{\text{ghost}} &= \partial_\mu \eta^{a\dagger}\left(\partial^\mu \delta^{ab} + g_s f_{abc} A^{c\mu}\right)\eta_b
\end{aligned}$$

- The gauge-fixing term is needed because of a degeneracy of sets of gluon field configurations that enter the path-integral formulation of QCD and which are equivalent under gauge transformation.

  $\hookrightarrow$ This degeneracy makes it impossible to write a gluon propagator. Adding gauge-fixing term to th Lagrangian solves the problem.

  $\hookrightarrow$ Choice of the parameter ξ fixes the gauge.
- On top of that, non-abelian gauge theory needs unphysical degrees of freedom, called ghosts, η, which are complex scalar fields obeying Fermi statistics.

Ways to solve QCD

When coupling is small $g_s \ll 1$:

- Perturbative expansion

$$\sigma = \underbrace{\sigma^{(1)} g_s}_{\text{leading order (LO)}} + \underbrace{\sigma^{(2)} g_s^2}_{\text{next-to-leading order (NLO)}} + \underbrace{\sigma^{(3)} g_s^3}_{\text{NNLO}} + \dots$$

  + \+ provides very precise results at high energies
  + – relies on $\sigma^{(i)}$ being all of the same order: not always true!
  + – unable to study QCD in the range of the proton mass $\sim 1\,\text{GeV}$

In principle, for any value of g_s:

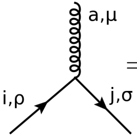
- Lattice QCD
  Put quarks and gluons on 4D-lattice and compute which configurations are most likely.
  + \+ excellent at calculating static properties like hadron masses
  + – only limited lattice sizes (hence large spacings) can be used in practice because of very high computational costs
  + – unable to address questions in collider physics because of missing analytic continuation from the imaginary to the real time

Feynman rules

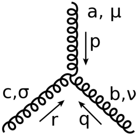
$$= \delta^{ab}\left[-g^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2+i\epsilon}\right]\frac{i}{p^2+i\epsilon}$$

$$= \delta^{ij}\frac{i(\not{p}+m)_{\mu\nu}}{p^2-m^2+i\epsilon}$$

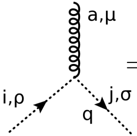
$$= \delta^{ab}\frac{i}{p^2+i\epsilon}$$



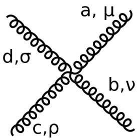
$$= -ig_s(T^a_{ji})_F\gamma^\mu_{\sigma\rho}$$



$$= -g_s f^{abc}\Big[(p-q)^\sigma g^{\mu\nu} + (q-r)^\mu g^{\nu\sigma} + (r-p)^\nu g^{\sigma\mu}\Big]$$



$$= g_s f^{aij} q^\mu$$



$$\begin{aligned} = & -ig_s^2 f^{xac} f^{xbd}\left[g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\gamma}\right] \\ & -ig_s^2 f^{xad} f^{xbc}\left[g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma}\right] \\ & -ig_s^2 f^{xab} f^{xcd}\left[g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\mu\rho}\right] \end{aligned}$$

Types of gauges

Covariant gauges: depend on a parameter ξ

$$= \quad \delta^{ab}\left[-g^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2+i\epsilon}\right]\frac{i}{p^2+i\epsilon}$$

Choices of ξ correspond to various gauges in this class:

- $\xi = 0$: Landau gauge
- $\xi = 1$: Feynman gauge

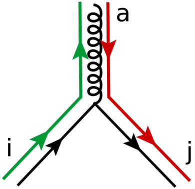
Axial gauges: depend on an arbitrary vector n_μ

$$= \quad \delta^{ab}\left[-g_{\mu\nu} + \frac{k_\mu n_\nu + k_\nu n_\mu}{k\cdot n} - \frac{n^2}{(k\cdot n)^2}k_\mu k_\nu\right]\frac{i}{p^2+i\epsilon}$$

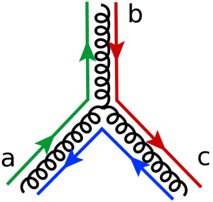
- big advantage: ghost contributions disappear
  $\hookrightarrow$ Faddeev-Popov determinant is A^a_μ-independent
- **Light-cone gauge**: a special case of axial-gauge with $n^2 = 0$
  $\hookrightarrow$ subtleties related to $k\cdot n$ singularities

Meaning of interactions

- Quarks carry colour and anti-colour, gluons carry colour-anti-colour
- Gluon repaints the quark as well as the gluon



$$-ig_s(T^a_{ji})_F\,\gamma^\mu_{\sigma\rho} = -ig_s(T^{\text{antigreen-red}}_{\text{redgreen}})_F\,\gamma^\mu_{\sigma\rho}$$

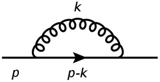


$$-g_s f^{abc}\left[(p-q)^\nu g^{\lambda\mu} + (p-q)^\nu g^{\lambda\mu} + (p-q)^\nu g^{\lambda\mu}\right]$$
$$= -g_s f^{\,\text{(green-antiblue) (antigreen-red) (antiblue-red)}}$$
$$\times\left[(p-q)^\nu g^{\lambda\mu} + (p-q)^\nu g^{\lambda\mu} + (p-q)^\nu g^{\lambda\mu}\right]$$

Renormalization

Let's calculate the quark self-energy graph in 4 dimensions

$$\int d^4 k$$

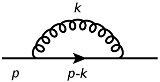


$$\sim \int^{k_{\text{cut}}} \frac{dk}{k} \sim \ln k_{\text{cut}}$$

Divergent as $k_{\text{cut}} \to \infty$: ultraviolet (UV) divergence.

The same in $D = 4 - 2\epsilon$ dimensions ($D < 4$, i.e. $\epsilon > 0$)

$$\int d^D k$$



$$= i \frac{\alpha_s}{4\pi} C_F \left(\frac{1}{\epsilon} + \ln(4\pi) - \gamma_E \right) \left(-3m + \xi(\not{p} - m) \right)$$

$$+ \text{ finite part}$$

UV divergence is a property of QCD (and many other QFTs):

- it arises because we extend our theory up to infinite energies, but each theory is valid only up to a certain scale Λ.

Renormalization

Divergences can be attributed a meaning and removed via the procedure of renormalization, which amounts to the following redefinitions

$$
\begin{aligned}
A^\mu &= Z_3^{1/2} A_R^\mu \\
q &= Z_2^{1/2} q_R \\
\eta &= \tilde{Z}^{1/2} \eta_R \\
g_s &= Z_g g_{sR} \mu^\epsilon \\
m^2 &= Z_m m_R^2
\end{aligned}
$$

- objects on the l.h.s. are the *bare* fields, *bare* coupling and *bare* mass, which we introduced in our original QCD Lagrangian
- on the r.h.s., we have the *renormalized*, *physical* fields, coupling and mass, which are finite and measurable in experiment

The Z_i coefficients contain divergences that cancel the divergences of the bare objects (A^μ, q, η, g_s, m^2) giving the finite renormalized objects (A_R^μ, q_R, η, g_{sR}, m_R^2).

Renormalization

The QCD Lagrangian (just the classical part for simplicity) takes the following form in terms of the renormalized fields

$$\begin{aligned}\mathcal{L}_{\text{classical}} &= Z_2 \bar{q}_R \left(i\partial\!\!\!/ - Z_m m_R\right) q_R - Z_2 Z_3^{1/2} Z_g g_{sR} \mu^\epsilon \bar{q}_R A\!\!\!/_R q_R \\ &\quad - \frac{Z_3}{4}\left(\partial_\mu A^a_{R\nu} - \partial_\nu A^a_{R\mu}\right)^2 - \frac{Z_3^2 Z_g^2 g_{sR}^2 \mu^{2\epsilon}}{4}\left(f^{abc} A^b_{R\mu} A^c_{R\nu}\right)^2 \\ &\quad + \frac{Z_3^{3/2} Z_g g_{sR} \mu^\epsilon}{2} f^{abc}\left(\partial^\mu A^{a\nu}_R - \partial^\nu A^{a\mu}_R\right) A^b_{R\mu} A^c_{R\nu}\end{aligned}$$

This can be rewritten as a sum of the original Lagrangian + counterterms

$$\begin{aligned}\mathcal{L}_{\text{classical}} &= \bar{q}_R \left(i\partial\!\!\!/ - m_R\right) q_R - g_{sR} \mu^\epsilon \bar{q}_R A\!\!\!/_R q_R + \ldots \\ &+ \bar{q}_R \Big((Z_2 - 1) i\partial\!\!\!/ - (Z_2 Z_m - 1) m_R \Big) q_R - (Z_2 Z_3^{1/2} Z_g - 1) g_{sR} \mu^\epsilon \bar{q}_R A\!\!\!/_R q_R + \ldots\end{aligned}$$

Hence, when doing computations one proceeds as follows

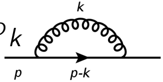
- use the Feynman rules discussed earlier (1^{st} line above, now with all objects renormalized)
- supplement that with the set of counterterm vertices (2^{nd} line above)

Renormalization

How do we get the Z_i coefficients?... By requiring the counterterm to cancel the pole part (PP) of Green functions.

For example, the quark self-energy graph gives:

$$\mathrm{PP} \int d^D k$$



$$= i \frac{\alpha_s}{4\pi} C_F S_\epsilon \left(-3m + \xi(\not{p} - m) \right),$$

where $S_\epsilon = \frac{1}{\epsilon} + \ln(4\pi) - \gamma_E$, and the corresponding counterterm:

$$= i \left[\not{p}(Z_2 - 1) - (Z_2 Z_m - 1)m \right].$$

Requirement of vanishing of the sum leads to the conditions:

$$i\not{p} \left[\frac{\alpha_s}{4\pi} C_F S_\epsilon \xi + Z_2 - 1 \right] = 0,$$

$$im \left[\frac{\alpha_s}{4\pi} C_F S_\epsilon (3m + \xi m) + (Z_2 Z_m - 1) \right] = 0,$$

and this fixes Z_2 and Z_m.

Renormalization: $\overline{\text{MS}}$ scheme

But wait! If we take e.g. the first condition with explicit S_ϵ

$$i\not{p}\left[\frac{\alpha_s}{4\pi}C_F\left(\frac{1}{\epsilon}+\ln(4\pi)-\gamma_E\right)\xi+Z_2-1\right]=0\,,$$

the Z_2 coefficient is fixed such that it cancels not only the pole $\frac{1}{\epsilon}$ but also the constant piece $\ln(4\pi)-\gamma_E$. Isn't that arbitrary? It is, and it is OK.

In the renormalization procedure, together with the infinite piece, one can subtract an arbitrary constant. That defines the renormalization scheme.
$\hookrightarrow$ Physical observables turn out not to depend on this arbitrary choice.

- Minimal Subtraction (MS) scheme: cancelling only the pole $\dfrac{1}{\epsilon}$
- Modified Minimal Subtraction ($\overline{\text{MS}}$) scheme: cancelling the pole $\dfrac{1}{\epsilon}$, together with the constant $\ln(4\pi)-\gamma_E$.

Z_i coefficients of the MS and the $\overline{\text{MS}}$ schemes are $\frac{m}{\mu}$ independent

- Z_is, by construction, cancel only the part singular at high momentum. But in this limit all masses are negligible and cannot appear in residues of the pole.

Renormalization

Applying similar procedure to other Green functions gives us the full set, of Zs, which, in $\overline{\mathrm{MS}}$, to the first order in α_s, read

$$
\begin{aligned}
Z_2 &= 1 - \frac{\alpha_s\, S_\epsilon}{4\pi\epsilon}\xi C_F + \mathcal{O}\left(\alpha_s^2\right) \\
Z_3 &= 1 - \frac{\alpha_s\, S_\epsilon}{4\pi\epsilon}\left[\left(\frac{\xi}{2} - \frac{13}{6}\right) C_A + \frac{4}{3} T_R n_f\right] + \mathcal{O}\left(\alpha_s^2\right) \\
\tilde{Z} &= 1 - \frac{\alpha_s\, S_\epsilon}{4\pi\epsilon} C_A \left(\frac{3}{4} - \frac{\xi}{4}\right) + \mathcal{O}\left(\alpha_s^2\right) \\
Z_m &= 1 - \frac{\alpha_s\, S_\epsilon}{4\pi\epsilon} 3 C_F + \mathcal{O}\left(\alpha_s^2\right) \\
Z_g &= 1 - \frac{\alpha_s\, S_\epsilon}{4\pi\epsilon}\left(\frac{11}{6} C_A - \frac{2}{3} T_R n_f\right) + \mathcal{O}\left(\alpha_s^2\right)
\end{aligned}
$$

Renormalization group

[To simplify notation, from now on, $g_s \to g_0$ (bare) and $g_{sR} \to g$ (renormalized)]

$$g_0 = g\mu^\epsilon Z_g = g\mu^\epsilon \left[1 - \frac{\alpha_s S_\epsilon}{4\pi\epsilon}\left(\frac{11}{6}C_A - \frac{2}{3}T_R n_f\right)\right]$$

The bare coupling cannot depend on μ, hence

$$0 = \frac{d}{d\ln\mu} g_0(\mu, g(\mu), \epsilon) = \epsilon g_0 + \frac{dg}{d\ln\mu}\frac{\partial g_0}{\partial g} \quad \Rightarrow \quad \frac{dg}{d\ln\mu} = -\frac{\epsilon g_0}{\frac{\partial g_0}{\partial g}}$$

In terms of

$$\alpha_s \equiv \frac{g^2}{4\pi}$$

we get

$$\beta(\alpha_s) \equiv \frac{d\alpha_s}{d\ln\mu^2} = \frac{g}{4\pi}\frac{dg}{d\ln\mu} = -\alpha_s^2\left[\frac{11C_A - 4T_R n_f}{12\pi} + \mathcal{O}(\alpha_s)\right]$$

- coupling runs with the scale μ^2
- $11C_A - 4T_R n_f = 21 > 0$, hence $\beta(\alpha_s) < 0$ in QCD

Running coupling

As $\beta(\alpha_s)$ is negative, α_s becomes small at high scales:
$\hookrightarrow$ **asymptotic freedom** [Gross, Wilczek, Politzer '73]

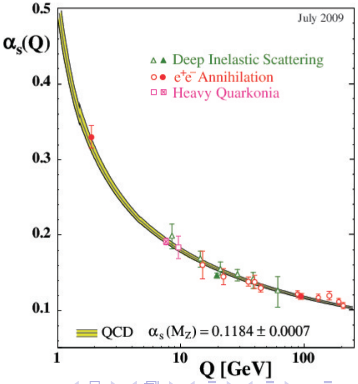
The renormalization group equation (here at lowest order):

$$Q^2 \frac{\partial \alpha_s}{\partial Q^2} = -b_0 \alpha_s^2 \quad \text{where} \quad b_0 = \frac{11 C_A - 4 T_R n_f}{12\pi}$$

allows one to relate couplings at two different scales Q_0 and Q

$$\alpha_s(Q^2) = \frac{\alpha_s(Q_0^2)}{1 + b_0 \alpha_s(Q_0^2) \ln \frac{Q^2}{Q_0^2}}$$

- at proton mass $\alpha_s(1\,\text{GeV}) \sim \mathcal{O}(1)$
- at the scale of the Z boson mass $\alpha_s(91\,\text{GeV}) \sim 0.1$



The Λ parameter

The one-loop running coupling diverges at low scales

$$\alpha_s(\mu^2) = \frac{\alpha_s(Q^2)}{1 + b_0\alpha_s(Q^2)\ln\frac{\mu^2}{Q^2}} \to \infty \qquad \text{as} \qquad 1 + b_0\alpha_s(Q^2)\ln\frac{\mu^2}{Q^2} = 0$$

Let us denote the scale at which this happens as $\mu^2 = \Lambda^2$. Solving the equation on r.h.s. above gives

$$\alpha_s(Q^2) = \frac{1}{b_0 \ln\frac{Q^2}{\Lambda^2}}$$

We've introduced the parameter Λ defined as the scale at which $\alpha_s = \infty$.

- $\Lambda \simeq 200\,\text{MeV}$ is measurable but it is not an observable as its value depends on: perturbative order, renorm. scheme, number of flavours.
- The order of magnitude of Λ indicates a scale at which α_s becomes large and perturbative theory is not applicable any longer.
- Notice that for massless QCD there is no mass scale in the theory as g_s is dimensionless. Mass scale however emerges via renormalization group and the appearance of Λ parameter (dimensional transmutation).

Exact symmetries of QCD

- Local gauge invariance.
- Baryon number conservation $B = \frac{1}{3}(n_q - n_{\bar{q}}) = \text{const.}$
- Discrete symmetries: charge conjugation (C), parity (P) and time reversal (T) invariance.
- There is one additional gauge invariant operator of mass dimension four that can be added to the Lagrangian

$$\mathcal{L}_\theta = \frac{\theta g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{\mu\nu}_a \quad \text{where} \quad \tilde{F}^{\mu\nu}_a = \frac{1}{2}\epsilon^{\mu\nu\sigma\rho} F_{a\,\sigma\rho}$$

  and this term violates CP symmetry.
  - This term is a total divergence so it does not contribute to perturbation theory (that is why it is absent in Feynman rules).
  - It turns out that due to non-trivial topological structure of the QCD vacuum it can however contribute via non-perturbative effects.
  - Experimental limit $\theta < 10^{-9}$. This raises the question: what makes it so small? - the so called strong CP problem. One popular solution is introduction of Axion particle.

Approximate symmetries of QCD

Isospin SU(2) symmetry

- $m_u \simeq 2.3\,\text{MeV}$, $m_d \simeq 4.8\,\text{MeV} \ll m_p$ hence u and d quarks have approx. equal masses and form a doublet of the SU(2) isospin group

Flavour SU(3) symmetry

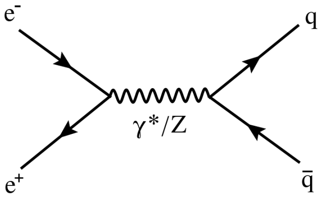
- adding the s quark, with $m_s \simeq 95\,\text{MeV}$ extends it to somewhat less accurate SU(3) flavour symmetry; representations of this group correspond to mesons and baryons and correctly predict the spectra

Chiral $SU(2)_L \otimes SU(2)_R$ symmetry

- For massless quarks, the left- and the right-handed fields decouple completely in the Lagrangian, which exhibits new *chiral* symmetry.
- Masses of u and d quarks are very small so the QCD Lagrangian shows approximate chiral symmetry.
- This symmetry is spontaneously broken at the level of the vacuum (which is non-trivial in QCD and connects left- and right-handed fields).
- This results in the appearance of three (number of broken generators) pseudo-Goldstone bosons: π^0, π^+ and π^-, which are indeed very light with $m \simeq 140\,\text{MeV}$ (they are not exactly massless as the chiral symmetry is not exact).

Infrared and collinear safety

Let's take the process $e^+e^- \to$ hadrons. At LO (*i.e.* Born level) we have



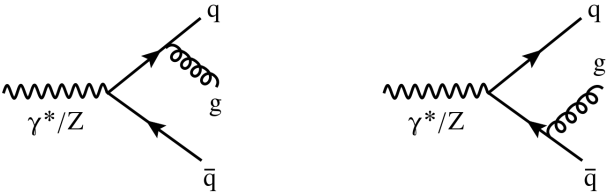
$$\sigma_{\text{Born}} = \frac{1}{\text{flux}} \int \sum |\mathcal{M}_{e^+e^- \to q\bar{q}}|^2 d\Phi_2 = \frac{4\pi\alpha_{\text{em}}^2}{3s} e_q^2 N_c$$

where s is the center-of-mass energy of the incoming e^+e^- pair and e_q is a quark charge.

- Factor N_c comes from sum over colours.

Infrared and collinear safety

NLO real correction to the $e^+e^- \to$ hadrons annihilation



$$|M_{\gamma\to q\bar{q}g}|^2 \propto \frac{q\cdot\bar{q}}{(q\cdot g)(\bar{q}\cdot g)} \sim \frac{1}{E_g^2}\frac{1}{(1-\cos\theta_{qg})}\frac{1}{(1-\cos\theta_{\bar{q}g})}$$

Hence

$$|M_{\gamma\to q\bar{q}g}|^2 \to \infty \quad \text{if} \quad \begin{cases} E_g^2 \to 0 & \text{infrared (soft) limit} \\ \theta_{qg} \to 0 & \text{collinear limit} \\ \theta_{\bar{q}g} \to 0 & \text{collinear limit} \end{cases}$$

- real correction to e^+e^- annihilation has soft the collinear divergences

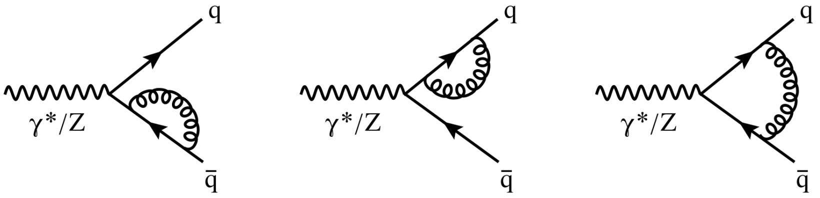
Infrared and collinear safety

The integral $\int |M_{\gamma\to q\bar{q}g}|^2 d\Omega$ can be performed in $D = 4 - 2\epsilon > 4$ dimensions and yields

$$\sigma_R^{e^+e^-\to q\bar{q}g} = \sigma_{\rm Born}\left\{\frac{\alpha_s}{2\pi}C_F\left(\frac{2}{\epsilon^2}+\frac{3}{\epsilon}+\frac{19}{2}\right)+\mathcal{O}(\epsilon)\right\}$$

- $\frac{1}{\epsilon^2}$ term corresponds to the soft divergence
- $\frac{1}{\epsilon}$ term corresponds to the collinear divergence
- $\frac{19}{2}$ is a finite term
- $\mathcal{O}(\epsilon)$ are terms vanishing in the limit $D \to 4$

Infrared and collinear safety: virtual correction



$$\sigma_V^{e^+e^-\to q\bar{q}} = M_{q\bar{q}}^{(\text{Born})} M_{q\bar{q}}^{(\text{virt})\,\dagger} = \sigma_{\text{Born}} \left\{ \frac{\alpha_s}{2\pi} C_F \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 \right) + \mathcal{O}(\epsilon) \right\}$$

- The same structure as in the case of real cross section: double and single poles in ϵ + regular term.

$e^+e^- \to$ hadrons: combined result

$$\begin{aligned}\sigma^{e^+e^-\to\text{hadrons}} &= \sigma_{\text{Born}} + \sigma_R^{e^+e^-\to q\bar{q}g} + \sigma_V^{e^+e^-\to q\bar{q}} \\ &= \sigma_{\text{Born}}\left\{1 + \frac{\alpha_s}{2\pi}C_F\left(\frac{2}{\epsilon^2}+\frac{3}{\epsilon}+\frac{19}{2}\right) + \frac{\alpha_s}{2\pi}C_F\left(-\frac{2}{\epsilon^2}-\frac{3}{\epsilon}-8\right) + \mathcal{O}(\epsilon)\right\} \\ &\overset{\epsilon\to 0}{=} \sigma_{\text{Born}}\left\{1 + \frac{\alpha_s}{2\pi}\frac{3}{4}C_F + \mathcal{O}\left(\alpha_s^2\right)\right\}\end{aligned}$$

- Collinear and soft divergences cancelled between real and virtual diagram emissions. We could safely take the $\epsilon \to 0$ limit.

Total cross section for e^+e^- annihilation to hadrons is a collinear and infrared safe observable.

- This is a manifestation of a more general theorem by Kinoshita, Lee and Nauenberg (KLN) and generalizations, which states that the soft and collinear singularities, present in real and virtual corrections, must cancel each other in the sum, for sufficiently inclusive observables.

Infrared and collinear safe observables

$$\frac{d\sigma}{dX} = \frac{1}{\text{flux}} \sum_n d\Phi^n |M^{(n)}|^2 \delta\big(X - f_X(p_1, \ldots, p_n)\big)$$

An observable X is called infrared and collinear safe if

$$f_X^{(n+1)}(p1, \ldots, p_n, p_{n+1}) \to \begin{cases} f_X^{(n)}(p1, \ldots, p_n) & \text{if} \quad p_{n+1} \to 0 \\ f_X^{(n)}(p1, \ldots, p_n + p_{n+1}) & \text{if} \quad p_n \parallel \ p_{n+1} \end{cases}$$

Physics behind this requirement:

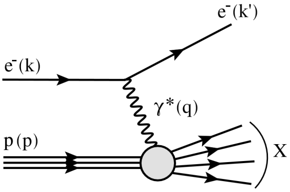
- one is not able to distinguish between configurations $|q\bar{q}\rangle$ and $\big|q\bar{q} + ng\,(\text{soft or collinear})\big\rangle$
- the results of measurements should not be dependent on detector's energy resolution and granularity

Summary of the features of QCD

- Quantum field theory with spin-$\frac{1}{2}$ quarks as fundamental degrees of freedom.
- The theory is asymptotically free: strength of interactions decreases with energy.
- Quarks carry new degree of freedom called colour.
- Number of colours $N_c = 3$.
- The theory exhibits local gauge invariance under $SU(3)$ colour group. This leads to appearance of gauge particles: the gluons.
- Quarks and gluons build bound states that are singlets of SU(3).

How do we know that QCD is the right theory?

Deep inelastic scattering (DIS) process



$$Q^2 = -q^2$$
$$x = \frac{Q^2}{2p\cdot q}$$
$$y = \frac{p\cdot q}{k\cdot p}$$

General form of the cross section

$$\frac{d^2\sigma}{dxdQ} = \frac{4\pi\alpha_{\text{em}}^2}{Q^4}\left\{\left[1+(1-y)^2\right]F_1(x,Q^2) + \tfrac{1-y}{x}\left[F_2(x,Q^2) - 2xF_1(x,Q^2)\right]\right\}$$

where F_1 and F_2 are the structure functions.

Hypothesis: proton consists of pointlike, spin-$\frac{1}{2}$, free objects called partons. $\gamma^* p$ interaction happens via γ^* interacting with exactly one parton. $\hookrightarrow$ That goes under the name of the parton model.

How do we know that QCD is the right theory?

Parton model hypothesis implies the so called Bjorken scaling

$$F_i(x, Q^2) \to F_i(x)$$

- If γ^* was scattering off non-pointlike constituents of size Q_0, F_i, which is dimensionless, would need to depend on the ratio Q/Q_0.

More specifically, in the parton model:

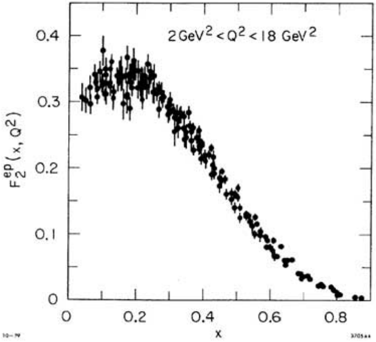
$$\begin{aligned} F_1(x) &= \frac{1}{2}\sum_i e_i^2 f_i(x)\,, \\ F_2(x) &= \sum_i e_i^2 x f_i(x)\,, \end{aligned}$$

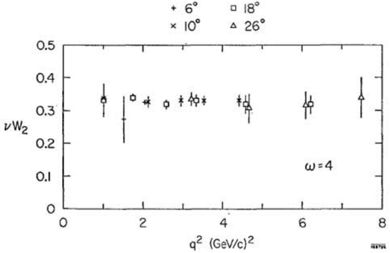
where $f_i(x)$, is a probability of finding a parton with momentum fraction x inside the proton, the so called parton density function (PDF).

- Seeing Bjorken scaling in the data would provide a strong evidence in favour of the parton model.

Bjorken scaling

And what does the data tell us? Back then in 1970...

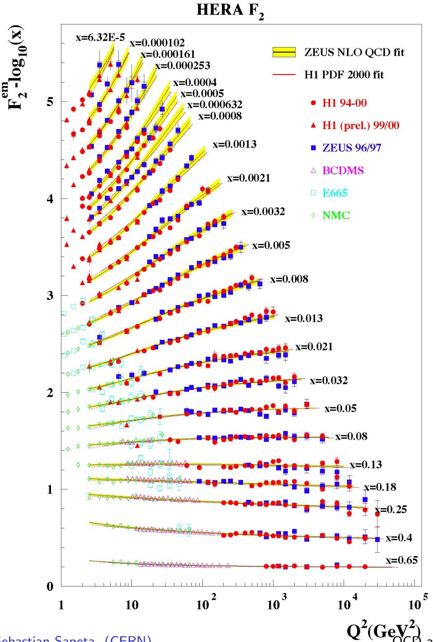




Conclusion: experimental evidence in favour of the parton model proves that the proton consists of objects that are

- pointlike ⇒ today we identify them with quarks
- free ⇒ that requires that the theory behind is asymptotically free

Bjorken scaling now



- data from DIS experiments: fixed target and HERA
- clearly visible region of Bjorken scaling for $x \gtrsim 0.1$
- we will come back to the region $x < 0.1$ in a minute

Callan-Gross relation

In the parton model

$$F_2(x) = 2xF_1(x) \qquad \text{(Callan-Gross relation)}$$

which follows from spin-$\frac{1}{2}$ property of partons.

One can construct the longitudinal structure function, corresponding to the absorption of the longitudinally polarized photons

$$F_L(x) = F_2(x) - 2xF_1(x)\,.$$

Callan-Gross relation means that $F_L = 0$ in the parton model.

- Follows from the fact that spin-$\frac{1}{2}$ parton cannot absorb a longitudinally polarized photon.
- In the experiment, we indeed see that F_L is very small. That confirms that partons are spin-$\frac{1}{2}$ particles.

Colour

Spin-statistics

The wave function of particles like like Δ^{++}:

$$|\Delta^{++}; +\frac{3}{2}\rangle = |u\uparrow\rangle|u\uparrow\rangle|u\uparrow\rangle$$

is totally symmetric in spin and flavour. That violates Pauli-principle unless there is an addition degree of freedom, in which the wave function is fully anti-symmetric. This is colour.

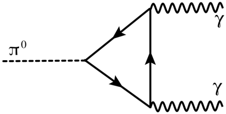
SU(3) color group leads to "white" baryons, *i.e.* hadrons and mesons are singlets of SU(3). Coloured particles are never observed: confinement.

- All that is consistent with experiment!

But how many colours?

$N_c = 3$

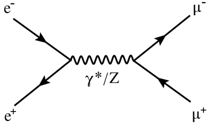
- $\pi^0 \to \gamma\gamma$ decay rate

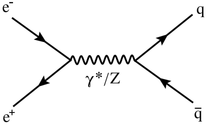


$$\Gamma(\pi^0 \to \gamma\gamma) = 7.63\,\mathrm{eV} \left(\frac{N_c}{3}\right)^2$$

Experimental value: $\Gamma(\pi^0 \to \gamma\gamma) = 7.84 \pm 0.56\mathrm{eV}$.

- e^+e^- decay ratio





$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_c\,\Sigma_q e_q^2 = N_c \frac{11}{9}$$

Experimental value: $N_c \simeq 3.2$.

What about gluons?

- Electron-nucleus DIS allows us to measure the momentum weighted probability density of quarks and anti-quarks in the nucleon

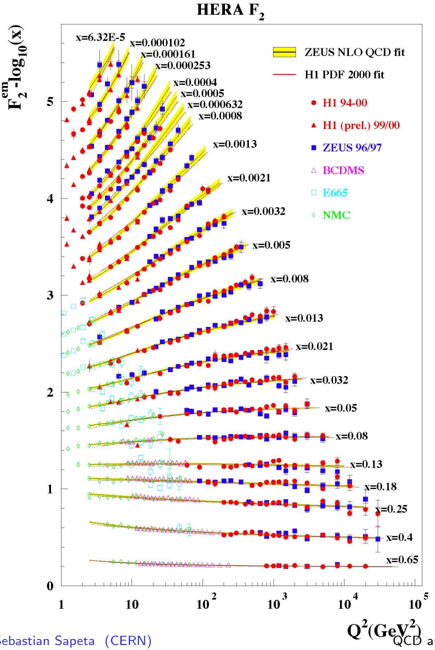
$$\frac{18}{5}\int_0^1 dx\, F_2^{eN}(x) = \int_0^1 dx\, x[u(x)+d(x)+\bar{u}(x)+\bar{d}(x)] \simeq 0.5$$

Charged particles carry only half of proton's momentum!

- Bjorken scaling holds only approximately and F_2 starts to depend on Q^2 as we go to lower values of x
  - this happens because of gluons which are produced in abundance at low-x
  - gluons go beyond the simple picture of the naive parton model

Violation of Bjorken scaling is indeed seen in the data at low-x!

Scaling violation seen in the data!



$$\lambda(x, Q^2) = -\frac{\partial F_2(x, Q^2)}{\partial \ln x}\bigg|_{Q^2}$$

$$F_2(x, Q^2) \simeq c(Q^2)\, x^{-\lambda(Q^2)}$$

From HERA data:

$\lambda \simeq 0.2 - 0.4$ for the range $x < 0.01$ and $Q^2 > 10\,\text{GeV}^2$

- large x: proton consists mostly of valence quarks and looks like in the naive parton model
- small x: proton consist mostly of gluons and the parton model needs to be improved with QCD

It all fits!

- Quantum field theory with spin-$\frac{1}{2}$ quarks as fundamental degrees of freedom : Callan-Gross relation, parton model ✓
- The theory is asymptotically free: strength of interactions decreases with energy: Bjorken scaling ✓
- Quarks carry new degree of freedom called colour: Δ^{++} & others ✓
- Number of colours $N_c = 3$: e^+e^- decay ratio & others ✓
- The theory exhibits local gauge invariance under SU(3) colour group. This leads to appearance of gauge particles: the gluons: scaling violation of F_2 ✓
- Quarks and gluons build bound states that are singlets of SU(3): coloured particles never observed ✓

\+ Countless tests from several generations of experiments!
(some of them covered in lectures 2 and 3)

Summary of lecture 1

- QCD is an extremely successful theory of strong interactions.
- It is based on SU(3) local color symmetry.
- It is a non-abelian theory which results in gluon self-interactions.
- In this lecture, we have concentrated on perturbative methods, which are applicable in the limit of small coupling.
- QCD exhibits UV divergence. It is removed by renormalization.
- Renormalization of QCD leads to running of the coupling α_s and the β function turns out to be negative, hence the strength of the interaction decreases with scale: asymptotic freedom.
- QCD exhibits also soft (infrared) and collinear divergencies.
- For correctly defined observables, soft and collinear divergences cancel between real and virtual contributions.
- In certain limit, QCD predicts Bjorken scaling of the DIS F_2 function and the violation of Bjorken scaling in another limit.
- QCD has withstood an enormous number of experimental tests.