Inflation
and
cosmological perturbations

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Astroparticules
et Cosmologie

Outline

1. **Homogeneous inflation**

2. **Cosmological perturbations**: from quantum fluctuations to observations

3. **Beyond the simplest models**

More details can be found in
• Lectures on inflation and cosmological perturbations, **arXiv:1001.5259 [astro-ph.CO]**
• Inflation, quantum fluctuations and cosmological perturbations (Cargese lectures) **arXiv:hep-th/0405053**

Standard cosmological model

- **General relativity:** $G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$

- **FLRW geometry** (spatial homogeneity and isotropy)

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$$

- **Matter:** $T^{\mu}_{\nu} = \mathrm{Diag}(-\rho, P, P, P)$

- **Friedmann equations**

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2}$$

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3P)$$

Cosmological evolution

- Three regimes with different eqs of state $P = w\,\rho$
  1. Radiation dominated regime ($w = 1/3$)
  2. Matter dominated regime ($w = 0$)
  3. Dark energy dominated regime ($w = -1$)

- Evolution of the scale factor (for $w \neq -1$)

$$\dot{\rho} + 3H(\rho + P) = 0 \qquad \Longrightarrow \qquad \rho \propto a^{-3(1+w)}$$

$$a(t) \propto t^{q} \quad \text{with} \quad q = \frac{2}{3(1+w)} < 1 \quad (w \equiv P/\rho = const)$$

- The Hot Big Bang model has been very successful but leaves several puzzles unsolved…

Flatness problem

- Deviation from flatness:

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2} \quad \Rightarrow \quad \Omega - 1 \equiv \frac{8\pi G\rho}{3H^2} - 1 = \frac{\kappa}{a^2H^2}$$

- Assuming a small curvature term initially

$$H^2 \propto \rho \propto a^{-3(1+w)} \quad \Rightarrow \quad (aH)^{-2} \propto a^{1+3w}$$

$w > -1/3 \Rightarrow |\Omega - 1|$ increases with time !

- Today, $|\Omega - 1| \lesssim 10^{-2}$. $|\Omega - 1|$ must have been extremely small in the past !

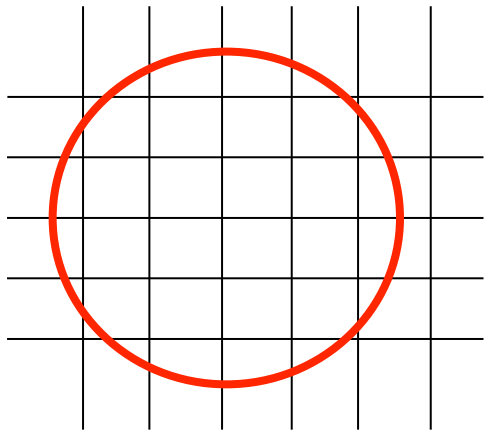
$$|\Omega_{\text{nucl}} - 1| < \mathcal{O}(10^{-16})$$

Horizon problem

- Horizon = maximum distance covered by a particle

For a radial light ray $dr = \dfrac{dt}{a(t)}$ $\left[ds^2 = -dt^2 + a^2(t)\left(dr^2 + \dots\right)\right]$

$$\lambda_{\rm hor}(t_i;t) = \int_{t_i}^{t} \frac{dt'}{a(t')} = \int_{a_i}^{a(t)} \frac{d\ln a}{aH}$$



comoving space

- Using $aH \propto a^{-(1+3w)/2}$

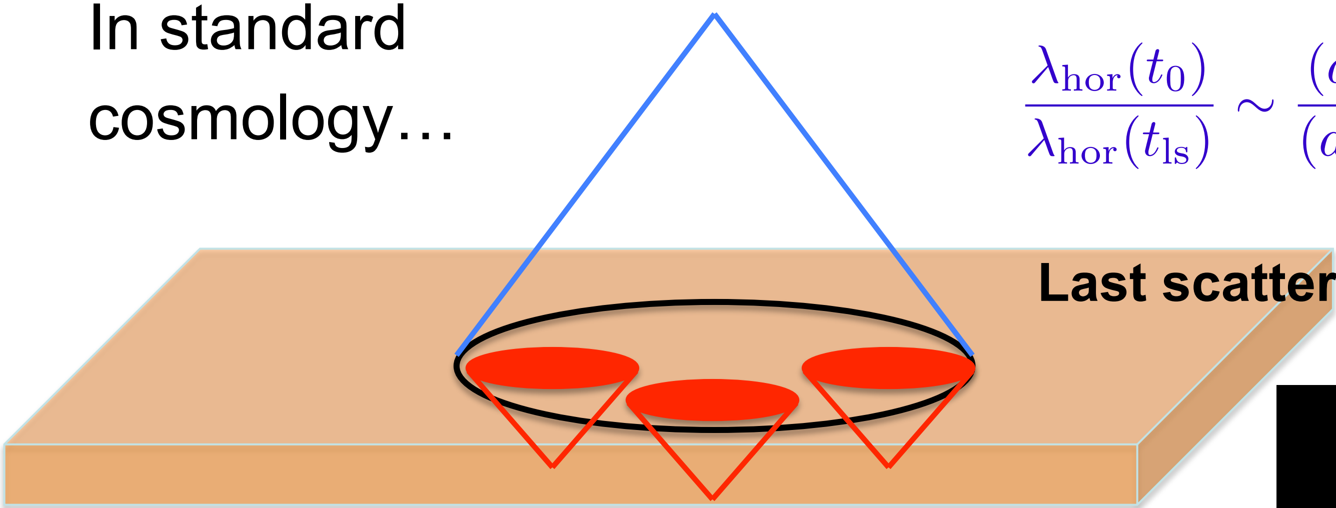
$$\lambda_{\rm hor} = \frac{2}{1+3w}(aH)^{-1}\left[1 - \left(\frac{a_i}{a}\right)^{\frac{1+3w}{2}}\right]$$

- If $w > -1/3$, this is finite, with $\lambda_{\rm hor} \sim (aH)^{-1}$

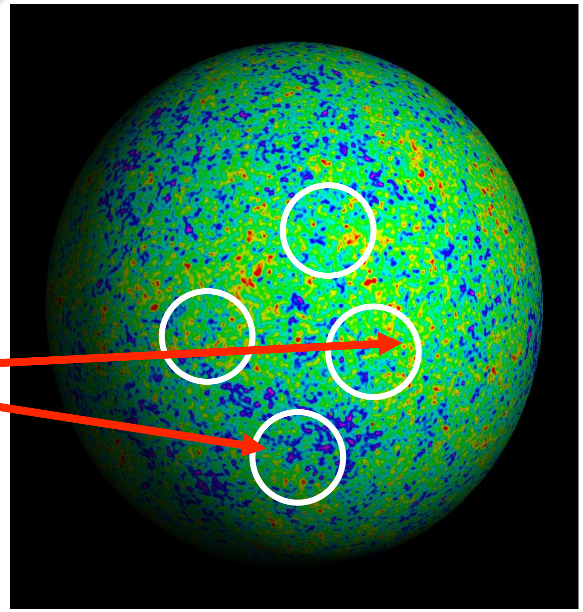
Horizon problem

In standard cosmology…

$$\frac{\lambda_{\rm hor}(t_0)}{\lambda_{\rm hor}(t_{\rm ls})} \sim \frac{(a_0 H_0)^{-1}}{(a_{\rm ls} H_{\rm ls})^{-1}} \sim \left(\frac{a_0}{a_{\rm ls}}\right)^{1/2}$$



Causally disconnected regions ?



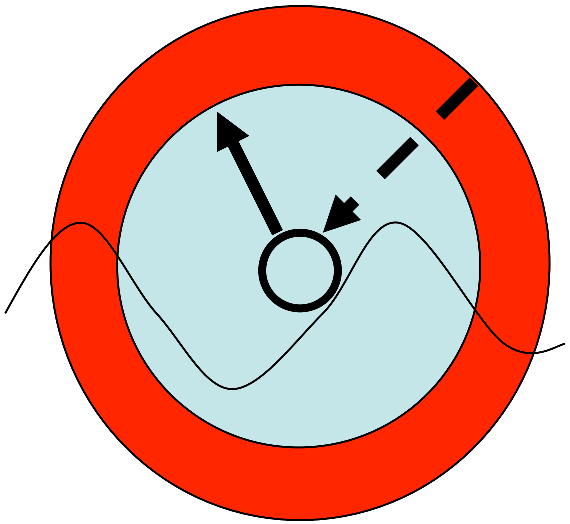
How to explain the quasi-isotropy of the CMB ? $\frac{\delta T}{T} \sim 10^{-5}$

Inflation

- A period of acceleration in the early Universe

$$\ddot{a} > 0 \Leftrightarrow (aH)^{-1}\text{decreases}$$

The horizon can be much bigger than the Hubble radius.



- Inflation also solves the flatness problem .

$$\Omega - 1 = \frac{\kappa}{a^2 H^2}$$

- Inflation also provides an explanation for the origin of the primordial perturbations, which will give birth to structures in the Universe.

Scalar field inflation

- How to get inflation ? $$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$$

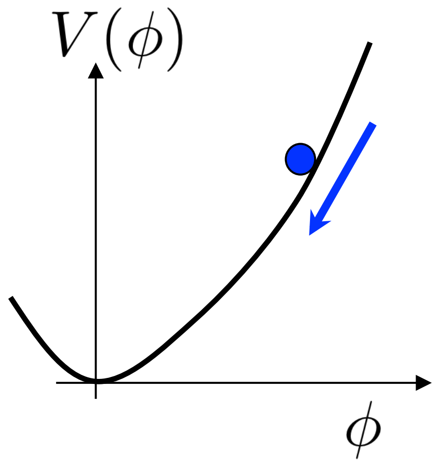
- Cosmological constant : $P = -\rho$ but no end…

- **Scalar field**

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad P = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

in a homogeneous universe

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0$$



Slow-roll regime

$$\frac{1}{2}\dot{\phi}^2 \ll V(\phi), \quad \ddot{\phi} \ll 3H\dot{\phi} \qquad\qquad P \simeq -\rho$$

- **Slow-roll equations**

$$H^2 \simeq \frac{8\pi G}{3}V \qquad\qquad 3H\dot{\phi} + V' \simeq 0$$

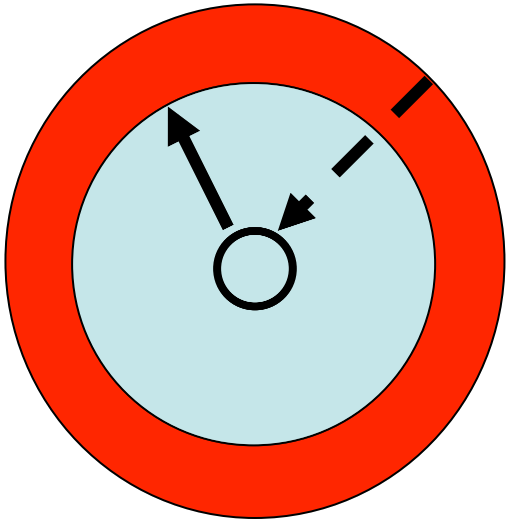
- **Slow-roll parameters**

$$\epsilon \equiv \frac{m_P^2}{2}\left(\frac{V'}{V}\right)^2 \ll 1 \qquad \eta \equiv m_P^2\frac{V''}{V} \ll 1 \qquad m_P \equiv \frac{1}{\sqrt{8\pi G}}$$

- **Number of e-folds** $N = \ln\frac{a_{end}}{a}$

$$dN = -d\ln a = -Hdt = -\frac{H}{\dot{\phi}}d\phi \qquad\qquad N(\phi) \simeq \int_{\phi}^{\phi_{end}} \frac{V}{m_P^2 V'}d\phi$$

How much inflation ?



- Inflation must last long enough to solve the horizon problem.

- Radiation era: the (comoving) Hubble radius $(aH)^{-1}$ increases like a

- Slow-roll inflation: $(aH)^{-1}$ decreases like a

$$\ln \frac{a_0}{a_{end}} \simeq \ln \frac{T_{end}}{T_0} \simeq \ln(10^{29} \frac{T_{end}}{10^{16}\mathrm{GeV}})$$

$$T_0 \sim 10^{-4}\mathrm{eV}$$

- Taking into account the matter dominated era

$$\ln \frac{a_0}{a_{end}} \simeq \ln(10^{27}) \simeq 62$$

Example 1

- Potential of the form $$V(\phi) = \frac{1}{2}m^2\phi^2$$

- Slow-roll: $\phi \gg \; m_P$

$$\epsilon \equiv \frac{m_P^2}{2}\left(\frac{V'}{V}\right)^2 = 2\frac{m_P^2}{\phi^2} \qquad \eta \equiv m_P^2\frac{V''}{V} = 2\frac{m_P^2}{\phi^2}$$

- Integration of the slow-roll eqs of motion yields

$$\phi - \phi_* = -\sqrt{\frac{2}{3}}\, m_P\, m(t - t_*)$$

- Using $$\frac{d\ln a}{d\phi} = \frac{H}{\dot{\phi}} \simeq -\frac{3H^2}{V'} \simeq -\frac{V}{m_P^2 V'} = -\frac{\phi}{2m_P^2}$$

  one finds

$$a = a_{end}\exp\left[-\frac{\left(\phi^2 - \phi_{end}^2\right)}{4\, m_P^2}\right] \qquad \Rightarrow \qquad N(\phi) \simeq \frac{\phi^2}{4m_P^2}$$

Example 2

- Exponential potential: $V(\phi) = V_0 \exp\left[-\sqrt{\frac{2}{q}}\frac{\phi}{m_P}\right]$

- Exact solution

$$a(t) \propto t^q, \qquad \phi(t) = \sqrt{2q}\, m_P \ln\left[\sqrt{\frac{V_0}{q(3q-1)}}\frac{t}{m_P}\right]$$

- Slow-roll ($q \gg 1$)

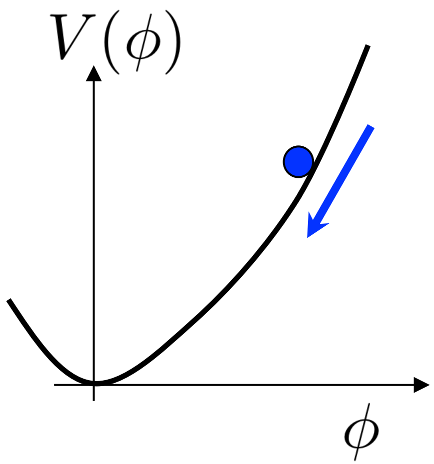
$$\epsilon \equiv \frac{m_P^2}{2}\left(\frac{V'}{V}\right)^2 = \frac{1}{q} \qquad\qquad \eta \equiv m_P^2 \frac{V''}{V} = \frac{2}{q}$$

Main families of inflation models

- Huge number of models of inflation
- Three main categories…

- **Large field models**

$\phi \sim m_P$ initially.



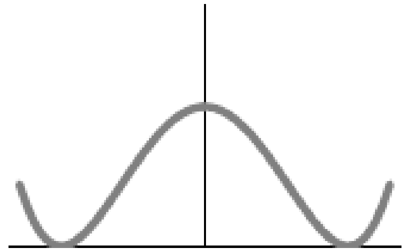
The scalar field rolls down toward
the minimun of the potential.

Typically $V(\phi) = V_0\, \phi^n$

Main families of inflation models

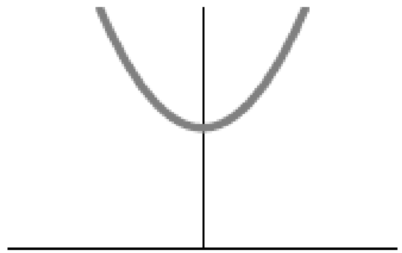
- **Small field models**

The initial value of the scalar field is small, typically near a maximum.



$$V(\phi) \simeq V_0 \left[1 - \left(\frac{\phi}{\mu}\right)^p\right]$$

- **Hybrid models**



$$V(\phi) \simeq V_0 \left[1 + \left(\frac{\phi}{\mu}\right)^p\right]$$

Requires a second field to end inflation

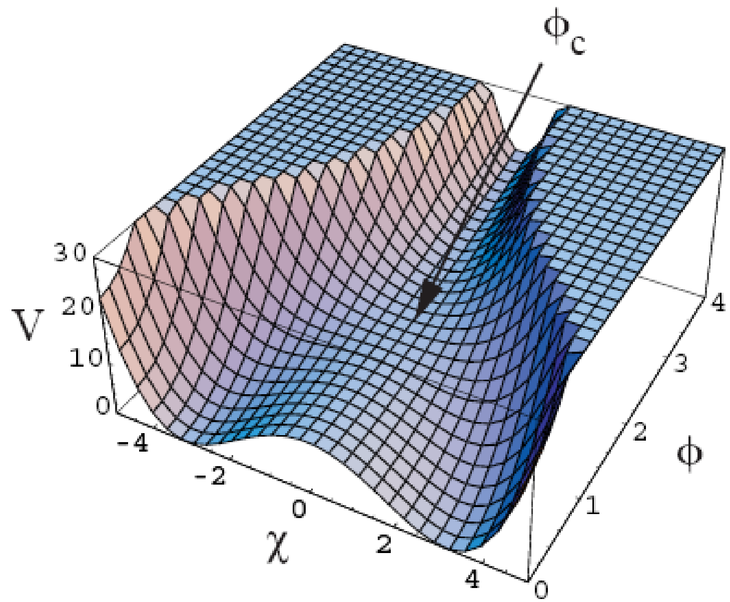
Hybrid inflation

- Potential: $V(\phi,\chi) = \frac{1}{2}m^2\phi^2 + \frac{1}{2}\lambda'\chi^2\phi^2 + \frac{1}{4}\lambda\left(\chi^2 - M^2\right)^2$

- Effective mass: $m_\chi^2 = -\lambda M^2 + \lambda'\phi^2$

- If $\phi > \phi_c \equiv \sqrt{\frac{\lambda}{\lambda'}}M$

  then $V_{\text{eff}}(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda M^4$



- If $\phi < \phi_c$, end of inflation…

Higgs inflation

- Non-minimal coupling of the Higgs to gravity: $\mathcal{L} = \xi H^\dagger H R$

$$S = \int d^4x \sqrt{-g} \left[\frac{M_P^2 + \xi h^2}{2} R - \frac{1}{2} \partial_\mu h \partial^\mu h - \frac{\lambda}{4} \left(h^2 - v^2 \right)^2 \right]$$

- From the Jordan frame to the Einstein frame

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \qquad \Omega^2 \equiv 1 + \frac{\xi h^2}{M_P^2}$$

$$S_E = \int d^4x \sqrt{-\tilde{g}} \left[\frac{M_P^2}{2} \tilde{R} - \frac{1}{2} \partial_\mu \chi \partial^\mu \chi - V(\chi) \right]$$

with the new field χ defined by $\frac{d\chi}{dh} = \sqrt{\frac{\Omega^2 + 6\xi^2 h^2 / M_P^2}{\Omega^4}}$

Higgs inflation

- Potential $$V(\chi) = \frac{1}{\Omega(\chi)^4} \frac{\lambda}{4} \left(h(\chi)^2 - v^2 \right)^2$$
- For $h \gg M_P/\sqrt{\xi}$ $$V(\chi) \simeq \frac{\lambda M_P^4}{4\xi^2} \left[1 + \exp\left(-\frac{2\chi}{\sqrt{6} M_P} \right) \right]$$

  The potential is very flat: $$\epsilon \simeq \frac{4M_P^4}{3\xi^2 h^4}, \qquad \eta \simeq -\frac{4M_P^2}{3\xi h^2}$$
- Number of e-folds before the end of inflation $$N(h) \simeq \frac{6}{8} \frac{h^2 - h_{\mathrm{end}}^2}{M_P^2/\xi} \qquad h_{\mathrm{end}} \simeq M_P/\sqrt{\xi}$$
- Observational constraints $$\frac{\xi}{\sqrt{\lambda}} \simeq 47 \times 10^3$$