Angular Momentum in Light-Front Dynamics

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Outline

Motivation

Light-front dynamics (LFD has been promoted for the study of bound states in quantized field theories, see for instance Brodsky, Pauli, and Pinsky, Phys. Rept. **301**, 299 (1998).

The main reasons why LFD is seen as a promising approach to QCD are as follows:

- In light-front (LF) quantization the greatest number of Poincaré generators, 7 out of 10, are kinematical, i.e., free of interaction;
- The kinematical boosts are sufficient to define any finite four momentum;
- The kinematical boosts together with the kinematical angular momentum operator J_z, generate a subgroup of the Poincaré group, the kinematical subgroup;

Motivation

- In LFD one can factorize out the dependence on the wave functions on the overall four momentum. The internal LF wave functions are invariant under kinematical Poincaré transformations;
- In LF gauge, $A^{+} = 0$, QCD is ghost-free;
- If one works in the LF gauge, the LF wave functions are also gauge invariant;
- Because not all rotations are kinematical, LFD has the disadvantage that rotational invariance is not manifest.
  In LF perturbation theory it is known how to obtain Lorentz invariant or covariant in LFD. In a non-perturbative situation, there are many open questions.

Poincaré Group

Generators of space-time transformations

P^{μ}	space-time translations
$M^{\mu\nu}$	pure Lorentz transformations

Commutation relations determine the Poincaré algebra, i.e., Poincaré group locally

$$
\begin{aligned}
[P^{\mu}, P^{\nu}] &= 0 \\
[M^{\mu\nu}, P^{\sigma}] &= i\,(P^{\mu} g^{\nu\sigma} - P^{\nu} g^{\mu\sigma}) \\
[M^{\mu\nu}, M^{\rho\sigma}] &= i\,(g^{\nu\rho} M^{\mu\sigma} - g^{\mu\rho} M^{\nu\sigma} + g^{\mu\sigma} M^{\nu\rho} - g^{\nu\sigma} M^{\mu\rho})
\end{aligned}
$$

Physical interpretation

$$
\begin{aligned}
\text{Angular momentum} \quad J^{i} &= \tfrac{1}{2}\epsilon_{ijk} M^{jk} \\
\text{Boosts} \quad K^{i} &= M^{0i}
\end{aligned}
$$

Forms of Relativistic Dynamics

Dirac (1949)

> *General relativity requires that physical laws expressed in terms of a system of curvilinear coordinates in space-time, shall be invariant under transformations from one such coordinate system to another.*

> *A second general requirement for dynamical theory has been brought to light through the discovery of quantum mechanics by Heisenberg and Schrödinger, namely the requirement that the equations of motion shall be expressible in the Hamiltonian form.*

These requirements limit the possible forms of dynamics, which involves a specification of the interactions.

In nonrelativistic dynamics there is only one dynamical generator, the Hamiltonian, and a unique way to write it:

$$H = \frac{\mathbf{p}^2}{2m} + V$$

The symmetry group is the Galilei group.

The state is specified by the initial conditions at a hypersurface $t = 0$, an *instant*.

For the Galilei group, an instant is the only appropriate initial surface.

In Einstein relativity, more possibilities are open: hypersurfaces Σ in Minkowski-space that do not contain timelike directions.

To simplify Dirac's homework assignment, one tries to find surfaces with the highest symmetries.

It amounts to finding the *stability group* G_Σ that maps Σ onto itself.

kinematical operators	generate elements of G_Σ
dynamical operators	map Σ into another hypersurface.

Transitivity

$$\forall x, y \in \Sigma : \exists\, g \in G_\Sigma \to x = gy,$$

This means that any point in Σ is connected to any other point by an element of the stability group.

There exist five different – inequivalent – possibilities, corresponding to five subgroups of the Poincaré group.

Dirac (1949):

$$\begin{aligned}
\text{Instant Form :} \quad & \Sigma = \{x^0 = 0\}, \\
\text{Point Form :} \quad & \Sigma = \{x^2 = a^2 > 0,\ x^0 > 0\}, \\
\text{Front Form :} \quad & \Sigma = \{x^0 + x^3 = 0\}
\end{aligned}$$

Leutwyler and Stern (1978)

$$\begin{aligned}
\Sigma &= (x^0)^2 - (x^1)^2 - (x^2)^2 = a^2 > 0,\ x^0 > 0, \\
\Sigma &= (x^0)^2 - (x^3)^2 = a^2 > 0,\ x^0 > 0
\end{aligned}$$

I shall only discuss the instant form (IFD) and the front form (LFD).

Dirac identified the "real difficulty" to be the specification of the interactions in each form of dynamics that is consistent with the Poincaré algebra.

Comparison of the Instant, Front, and Point Forms

We use the notation:

$$A^{\pm} = \frac{1}{\sqrt{2}}(A^0 \pm A^3), \quad \mathbf{A}_{\perp} = (A^1, A^2),$$

$$A \cdot B = A_\mu B^\mu = A^- B^+ + A^+ B^- - \mathbf{A}_\perp \cdot \mathbf{B}_\perp$$

In the instant form one introduces the four-velocity u^μ. Then the momentum is given by

$$p^\mu = m\, u^\mu, \quad u^\mu u_\mu = 1$$

instant form

rotations and three-momenta are kinematical

the energy and the boosts are dynamical

point form

rotations and boosts kinematical

all components of four-momentum dynamical

front form

some rotations and boosts are kinematical

the others are dynamical

the z-direction is special

Instant Form

The dynamical generators are

$$
\begin{aligned}
P^0 &= \sum \sqrt{\mathbf{p}^2 + m^2} + V, \\
M^{0r} &= \sum x^r \sqrt{\mathbf{p}^2 + m^2} + V^r,
\end{aligned}
$$

V must be a three-dimensional scalar, independent of the origin of $\mathbf{x}$ and $\mathbf{V}$ must be a three-dimensional vector of the following form

$$\mathbf{V} = \mathbf{x}\, V + \mathbf{V}',$$

$\mathbf{V}'$ independent of the origin of the coordinates

The commutators $[V, \mathbf{V}]$, $[V^i, V^j]$ follow from the Poincaré algebra

Point Form

$$P^\mu = \sum [p^\mu + x^\mu B(p^2 - m^2)] + V^\mu,$$

where the function B is given by

$$B(p^2 - m^2) = \frac{1}{x^2}\left[\sqrt{(p \cdot x)^2 - x^2(p^2 - m^2)} - p \cdot x\right]$$

V^μ is a four-vector, that must satisfy the commutators $[V^\mu, V^\nu]$ that follow from $[P^\mu, P^\nu] = 0$

Front Form

$$P^- = \sum \frac{\mathbf{p}_\perp^2 + m^2}{2p^+} + V,$$

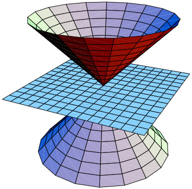
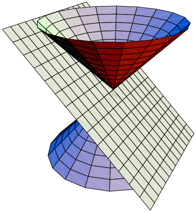
$$M^{-\,i} = \sum \left[x^i \frac{\mathbf{p}_\perp^2 + m^2}{2p^+} - x^- p^i \right] + V^i$$

V must be invariant under all transformations of $\mathbf{x}_\perp$ and x^-, except $x^- \to \lambda x^- \Rightarrow V \to \lambda V$

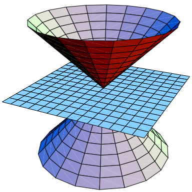
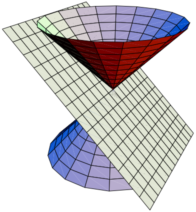
$$\mathbf{V}_\perp = \mathbf{x}_\perp V + \mathbf{V}'_\perp,$$

$\mathbf{V}'_\perp$ has the same limitations as V, and in addition must be a vector under rotations around the z-axis

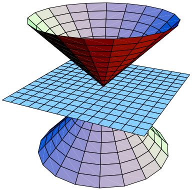
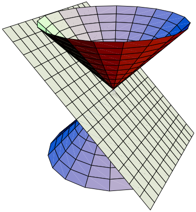
Summary: IFD and LFD Compared

Instant Form	**Front Form**
	
Quantization Surface	
$x^0 = 0$	$x^0 + x^3 = 0$

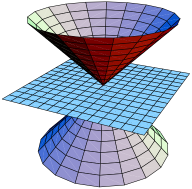
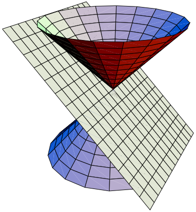
Summary of Forms of Dynamics, continued

Instant Form	**Front Form**
	
Kinematical Generators	
$\mathbf{P}$ $\mathbf{J}$	P^+, $\mathbf{P}_\perp$ $E^1 = M^{+1} = \frac{K_x + J_y}{\sqrt{2}}$ $E^2 = M^{+2} = \frac{K_y - J_x}{\sqrt{2}}$ $J_z = M^{12}$ $K_z = M^{-+}$

Summary of Forms of Dynamics, continued

Instant Form	**Front Form**
	
Dynamical Generators	
P^0 $\mathbf{K}$	P^- $F^1 = M^{-1} = \frac{K_x - J_y}{\sqrt{2}}$ $F^2 = M^{-2} = \frac{K_y + J_x}{\sqrt{2}}$

Summary of Forms of Dynamics, continued

Instant Form	**Front Form**
	
Plane-wave Representation	
$\lvert\mathbf{p}\rangle$ $p^0 = \pm\sqrt{\mathbf{p}^2 + m^2}$ $p^0 > 0$ and $p^0 < 0$	$\lvert p^+, \mathbf{p}_\perp\rangle$ $p^- = \frac{\mathbf{p}_\perp^2 + m^2}{2p^+}$ $p^- > 0 \leftrightarrow p^+ > 0$
Measure	
$\int \frac{d^3 p}{2p^0}$	$\int \frac{d^2 p_\perp dp^+}{2p^+}$

Bakamjian-Thomas Construction

Bakamjian and Thomas (1953) gave a complete construction, starting from an invariant mass operator and putting all interaction dependence solely through that mass operator.

$$
\begin{aligned}
P^0 &= \sqrt{\mathbf{p}^2 + M_{\mathrm{IF}}^2}, \quad \text{(instant form)},\\
P^\mu &= M_{\mathrm{PF}}\, u^\mu, \quad \text{(point form)}\\
P^- &= \frac{\mathbf{p}_\perp^2 + M_{\mathrm{LF}}^2}{2P^+} \quad \text{(front form)}
\end{aligned}
$$

The formal limitations set by relativistic invariance do not determine the interactions explicitly. One may resort to field theory.

Lorentz Transformations and Goodness

The generators of the Poincaré group can be classified with respect to their behaviour under boosts.
Consider a boost of the reference frame along the positive z-axis:

$$x'^{\mu} = B(\beta \mathbf{e}^3)^{\mu}_{\nu} \, x^{\nu}$$

with

$$B(\beta \mathbf{e}^3)^{\mu}_{\nu} = \begin{bmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma\beta \end{bmatrix}$$

One can write the boost as

$$B(\beta \mathbf{e}^3) = \exp(-\mathrm{i}\zeta_3 K_z)$$

with

$$\gamma = \cosh \zeta_3, \quad \gamma\beta = \sinh \zeta_3.$$

One defines the goodness g_A of an operator A as follows: if the commutator of A with K_z is proportional to A then

$$\mathrm{i}[K_z, A] = g_A A,$$

otherwise it is not defined. This concept is used to characterize the scaling of operators under a boost.
For an operator with goodness g_A one finds for the transformed operator

$$A(\zeta_3) = e^{-i\zeta_3 K_z} A e^{+i\zeta_3 K_z} = e^{-g_A \zeta_3} A.$$

Thus, we find that such operators scale with ζ_3 according to their goodness. The usual Poincaré generators do not have definite goodness, but by taking linear combinations, one finds operators that do have it:

$g = +1$	$g = 0$	$g = -1$
$P^+ = \frac{P^0 + P^3}{\sqrt{2}}$	$\mathbf{P}_\perp$	$P^- = \frac{P^0 - P^3}{\sqrt{2}}$
$E^1 = \frac{K_x + J_y}{\sqrt{2}}$	J_z	$F^1 = \frac{K_x - J_y}{\sqrt{2}}$
$E^2 = \frac{K_y - J_x}{\sqrt{2}}$	K_z	$F^2 = \frac{K_y + J_x}{\sqrt{2}}$

Infinite-Momentum Frame

If one boosts one's reference frame along the negative z-axis, $\beta < 0 \leftrightarrow \zeta_3 < 0$, the good operators are multiplied with e^{ζ_3}, which means that they dominate when ζ_3 is made large. If the limit $\zeta_3 \to \infty$ is taken, such a frame is called the infinite-momentum frame (IMF).

In LFD, the kinematical subgroup of the Poincaré group is generated by generators that have goodness either $+1$ or 0. For that reason, one sometimes identifies LFD with working in the IMF, which is not justified, because one can quantize a system in its restframe using light-front quantization.

Still, it turns out BLGB, Lecture Notes in Physics **572**, 1, (2001) that using LFD leads to the same simplifications as found by Weinberg, Phys. Rev. **150**, 1313 (1966). However, one should be aware of the fact that there exist treacherous points in LFD, BLGB, Few-Body Systems **49**, 177 (2011), that may be overlooked if one takes the infinite-momentum limit in a naive way. Some of those treacherous points are associated with the occurrence of singularities that are caused by the peculiar dispersion relation for the LF energy P^- that contains P^+ in the denominator.

Angular Momentum

In IFD one may easily construct the angular-momentum operators for free particles. One uses the Pauli-Lubansky vector W^μ defined as

$$W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\alpha\beta} P_\nu M_{\alpha\beta}.$$

Trivially, one notices that $W^\mu P_\mu = 0$

If one substitutes the physical form of the tensor elements one finds

$$W^0 = \mathbf{P} \cdot \mathbf{J}, \quad \mathbf{W} = P^0 \mathbf{J} - \mathbf{P} \times \mathbf{K}$$

This representation shows that in the rest frame of a particle with mass m, where $\mathbf{P} = 0$, the Pauli-Lubanski vector reduces to

$$\mathring{W}^\mu = (0, m\mathbf{J}) := (0, m\,\mathbf{s})$$

The vector $\mathbf{s}$ is known as the spin vector in the rest frame. De invariant $W^\mu W_\mu$ can be written in terms of the rest-frame spin as follows:

$$W^\mu W_\mu = -m^2\,\mathbf{s}^2$$

In a frame where the particle moves we write

$$W^\mu = m(S^0, \mathbf{S})$$

We may obtain the form written in Jackson (1999) by applying a pure boost to the rest-frame spinvector to obtain:

$$S^\mu = \left(\frac{\mathbf{p} \cdot \mathbf{s}}{m}, \mathbf{s} + \frac{(\mathbf{p} \cdot \mathbf{s})\mathbf{p}}{m(p^0 + m)} \right)$$

The components of W do not have definite goodness, but W^+ has, namely goodness $+1$.

By substitution of the physical matrix elements one finds

$$W^+ = P^+ J^3 + P^2 E^1 - P^1 E^2$$

We sea that W^+ is expressed in terms of kinematical generators.

Even more important is the fact that we can use this generator to construct an invariant for the kinematical subgroup.

Light-front spin operator

Define $\mathcal{J}^3$ as follows

$$\mathcal{J}^3 = \frac{W^+}{P^+}$$

It is a trivial matter to calculate the commutators of $\mathcal{J}^3$ with all kinematical generators, except K^3. The latter can be obtained by writing $W^+ = P^+\mathcal{J}^3$ and express the commutators of K^3 with W^+ in terms of its commutators with P^+ and W^+. Then one finds indeed that $[K^3, \mathcal{J}^3]$ vanishes, like the commutators of $\mathcal{J}^3$ with the other generators of the kinematical subgroup.

One may write

$$\mathcal{J}^3 = B(\zeta)J^3$$

where

$$B(\zeta) = e^{-i(\zeta_1 E^1 + \zeta_2 E^2)}e^{-i\zeta_3 K^3}$$

is the most general kinematical LF boost. Then one defines

$$\mathcal{J}^i = B(\zeta)J^i, \quad (i = 1, 2, 3)$$

The components $\mathcal{J}^1$ and $\mathcal{J}^2$ are not kinematical, thus in order to implement them in a concrete calculation, one must find an explicit representation in terms of the dynamics.

It is perhaps needless to say that presently it is not possible to find such a representation in QCD.

One can characterize states in strongly interacting systems by using the eigenvalues of the kinematical generators including $\mathcal{J}^3$, which is denoted as the LF helicity, h.

Thus one finds basis states, characterized by the kinematical components of the momentum and the LF helicity

$$|\mathbf{p}_{\mathrm{LF}}, h, n\rangle$$

where $\mathbf{p}_{\mathrm{LF}} = (p^+, p^1, p^2)$, h is the LF helicity, and n symbolizes the "internal quantum numbers", like color and flavour in QCD.

LF boost: details

Using the parametrization involving ζ_1, ζ_2, and ζ_3, we write the kinematical boost as follows:

$$B_{\mathrm{LF}}(\boldsymbol{\zeta}_\perp;\zeta_3)^\mu_\nu = \begin{pmatrix} e^{\zeta_3} & 0 & 0 & 0 \\ e^{\zeta_3}\zeta_x & 1 & 0 & 0 \\ e^{\zeta_3}\zeta_y & 0 & 1 & 0 \\ \frac{\boldsymbol{\zeta}_\perp^2}{2}e^{-\zeta_3} & \zeta_x e^{-\zeta_3} & \zeta_y e^{-\zeta_3} & e^{-\zeta_3} \end{pmatrix}, \ \mu,\nu \in \{+,1,2,-\}.$$

If $B_{\mathrm{LF}}(\boldsymbol{\zeta}_\perp;\zeta_3)$ boosts a particle with velocity 0 to one with finite velocity:

$$B_{\mathrm{LF}}(\boldsymbol{\zeta}_\perp;\zeta_3)\left(\frac{m}{\sqrt{2}},0,0,\frac{m}{\sqrt{2}}\right) = (p^+, p_x, p_y, p^-),$$

the following relations hold for the parameters to the momentum components:

$$\frac{m}{\sqrt{2}}e^{\zeta_3} = p^+,\ \frac{m}{\sqrt{2}}e^{\zeta_3}\boldsymbol{\zeta}_\perp = \mathbf{p}_\perp,\ \frac{m}{\sqrt{2}}\left(\frac{\boldsymbol{\zeta}_\perp^2}{2}+1\right)e^{-\zeta_3} = p^- = \frac{\mathbf{p}_\perp^2 + m^2}{2p^+}$$

One sees that p^- has the correct LF dispersion relation for the energy of a free particle of mass m.

The kinematical boosts are represented by *lower triangular* matrices, which form a group.

We see that compared to an IF boost, the LF boost performs a rotation, the Melosh rotation. This rotation can be found by writing the LF boost as

$$B_{\text{LF}}(\boldsymbol{\zeta}_\perp; \zeta_3) = B_{\text{IF}} R_{\text{M}}$$

where B_{IF} is a pure Lorentz boost and R_{M} is a pure rotation. Its form is rather complicated, but we can write the cosine of the rotation angle quite easily

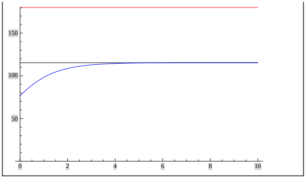
$$\theta_{\text{M}} = \arccos \frac{2(1+e^{\zeta_3})^2 - e^{2\zeta_3}\boldsymbol{\zeta}_\perp^2}{2(1+e^{\zeta_3})^2 + e^{2\zeta_3}\boldsymbol{\zeta}_\perp^2}$$

It is instructive to consider the limits $\zeta_3 \to \infty$ and $\boldsymbol{\zeta}_\perp^2 \to 0$.

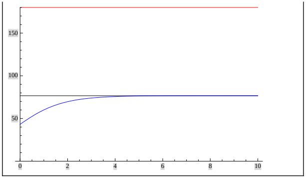
$$\lim_{\zeta_3 \to \infty} \cos\theta_{\text{M}} = \frac{2 - \boldsymbol{\zeta}_\perp^2}{2 + \boldsymbol{\zeta}_\perp^2}, \quad \lim_{\boldsymbol{\zeta}_\perp^2 \to 0} \cos\theta_{\text{M}} = 1.$$

We see that the boost along the z-axis involved in the kinematic boost makes $p_z > 0$ for positive ζ_3. The Melosh rotation can rotate the momentum to produce a negative p_z only if $\boldsymbol{\zeta}_\perp^2 > 2$.

We show the rotation angle and its asymptotic value (in black).



The rotation angle of the Melosh rotation for $\zeta_\perp = (2, 1)$ as a function of ζ_3. Because $\zeta_\perp^2 > 1$, the angle can become larger than 90°.



The rotation angle of the Melosh rotation for $\zeta_\perp = (1, 1/2)$ as a function of ζ_3. Because $\zeta_\perp^2 < 1$, the angle stays below 90°. For $\zeta_\perp^2 \to \infty$ the angle goes to 180°, which is a singular point of the kinematic LF boosts.

LF Spinors

LF spinors are constructed using the SL(2,C) LF kinematic boost on a spinor defined in its rest frame. If this spinor corresponds to spin projection on the z-axis equal to h, one finds for the boosted one

$$u_{\mathrm{LF}}(p,1/2) = \frac{1}{\sqrt{m\sqrt{2}p^+}}\left(p^+, \frac{p_x+ip_y}{\sqrt{2}}, \frac{m}{\sqrt{2}}, 0\right)$$

$$u_{\mathrm{LF}}(p;-1/2) = \frac{1}{\sqrt{m\sqrt{2}p^+}}\left(0, \frac{m}{\sqrt{2}}, \frac{p_x-ip_y}{\sqrt{2}}, p^+\right)$$

They are eigenvectors of the LF helicity

$$h_{\mathrm{LF}} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \frac{p_x+ip_y}{\sqrt{2}p^+} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{p_x-ip_y}{\sqrt{2}p^+} \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

LF Polarization vectors

LF polarization vectors are constructed using the 4D LF kinematic boost on a polarization vector defined in a frame where its three momentum is along the $+z$-axis and circulair polarization h. One then finds for a particle with mass m

$$\epsilon_{\mathrm{LF}}(p,\pm 1) = \left(0, \mp\frac{1}{\sqrt{2}}, \frac{-i}{\sqrt{2}}, \mp\frac{p_x \pm i p_y}{\sqrt{2}p^+}\right)$$

$$\epsilon_{\mathrm{LF}}(p,0) = \left(\frac{p^+}{m}, \frac{p_x}{m}, \frac{p_y}{m}, \frac{\mathbf{p}_\perp^2 - m^2}{2mp^+}\right)$$

They are eigenvectors of the LF helicity

$$H_{\mathrm{LF}} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \frac{p_x+ip_y}{\sqrt{2}p^+} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{p_x-ip_y}{\sqrt{2}p^+} \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

Fock Expansion

In LF field theory, a Fock-state expansion makes sense, because the perturbative vacuum is the true vacuum, there are no vacuum diagrams in LF perturbation theory. Thus we write for a state with momentum P:

$$|P^+, \mathbf{P}_\perp\rangle = \sum_n \Phi_n |p_1^+, \mathbf{p}_{1\perp}, \ldots, p_n^+, \mathbf{p}_{n\perp}\rangle$$

The LF kinematic momenta are denoted as $p_i^+, \mathbf{p}_{i\perp}$ thus giving

$$P^+ = \sum_i p_i^+, \quad \mathbf{P}_\perp = \sum_i \mathbf{p}_{i\perp}$$

Frequently, one works in the frame where $\mathbf{P}_\perp = 0$ and uses the fractions $x_i = p_i^+/P^+$, which gives

$$\sum_i x_i = 1, \quad \sum_i \mathbf{p}_{i\perp} = 0.$$

Relative Momentum

As an example we look at the case of two constituents, i.e., free particles with masses m_1 and m_2

$$P = p_1 + p_2 \Leftrightarrow P^+ = p_1^+ + p_2^+, \quad \mathbf{P}_\perp = \mathbf{p}_{1\,\perp} + \mathbf{p}_{2\,\perp}$$

If the invariant mass is denoted as $P^2 = M_0^2$, the energy-momentum dispersion relation is

$$P^- = \frac{\mathbf{P}_\perp^2 + M_0^2}{2P^+}.$$

Define relative variables

$$x = \frac{p_1^+}{P^+}, \quad \mathbf{q}_\perp = (1-x)\mathbf{p}_{1\,\perp} - x\mathbf{p}_{2\,\perp}.$$

Then

$$\mathbf{p}_{1\,\perp} = x\mathbf{P}_\perp + \mathbf{q}_\perp, \quad \mathbf{p}_{2\,\perp} = (1-x)\mathbf{P}_\perp - \mathbf{q}_\perp.$$

For free particles the invariant mass in terms of x and $\mathbf{q}_\perp$ is

$$M_0^2 = 2P^+P^- - \mathbf{P}_\perp^2 = \frac{\mathbf{q}_\perp^2 + m_1^2}{x} + \frac{\mathbf{q}_\perp^2 + m_2^2}{1-x}$$

For interacting particles, the interaction may be added to P^- or to M_0^2.

For the discussion of angular momentum it is relevant to define a vector $\mathbf{q}$, by adding q_z to $\mathbf{q}_\perp$, given by

$$q_z = x(1-x)\frac{\partial M_0}{\partial x} = (x - \tfrac{1}{2})M_0 - \frac{m_1^2 - m_2^2}{2M}.$$

Both the fraction x and the relative momentum squared are invariant under LF kinematic boosts. The expression for M_0 gives

$$\mathbf{q}_\perp^2 = x(1-x)M_0^2 - (1-x)m_1^2 - xm_2^2.$$

So we can calculate the square of $\mathbf{q}$:

$$\mathbf{q}_\perp^2 + q_z^2 = \frac{(M_0^2 - m_1^2 - m_2^2)^2 - 4m_1^2m_2^2}{4M_0^2}.$$

M_0 is invariant $\Rightarrow$ $\mathbf{q}^2$ invariant

Relative Angular Momentum

One can write the following relation

$$M_0 = \sqrt{m_1^2 + \mathbf{q}^2} + \sqrt{m_2^2 + \mathbf{q}^2} \equiv E_1 + E_2.$$

This suggests that $\mathbf{q}$ can be related to the orbital angular momentum operator $\mathbf{L}$

The orbital part of the LF helicity L_3 is given by

$$L_3 = -i\left(q^1 \frac{\partial}{\partial q^2} - q^2 \frac{\partial}{\partial q^1}\right),$$

One may define other components ($r = 1, 2$, $\epsilon_{12} = 1$, $\epsilon_{21} = -1$):

$$L_r = i\epsilon_{rs}\left[-2\frac{q^s}{M_0}\frac{\partial}{\partial x} + x(1-x)\frac{\partial M_0}{\partial x}\frac{\partial}{\partial q^s}\right],$$

to find

$$\mathbf{L} = -i\mathbf{q} \times \boldsymbol{\nabla}_q.$$

Caution

Despite appearences, the quantity $\mathbf{q}$ is not a true three vector in an interacting theory because it is defined in terms of LF kinematic variables only and thus the transformation of $\mathbf{q}$ is not guaranteed under dynamic transformations, like the rotations generated by $\mathbf{L}_\perp$.

If one would replace the free-particle invariant mass M_0 by the mass of the interacting system M, one needs to find this mass first, which means solve the whole theory, say QCD.

The best one can do is stick with the kinematical pieces of the angular momentum, i.e., the LF helicities.

Summary

- In light-front dynamics, one can safely calculate kinematical quantities of constituents.
- It does not make sense in LFD to discuss besides the kinematic components of the momenta and the angular momenta, which are conserved, also the dynamical ones in terms of constituents.
- It seems clear that what can be measured is the LF helicity. The purely formal treatment here does not give advise as to how the orbital and the intrisic part of the LF helicity could be measured separately.