Transverse Spin Sum Rules

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There are many transverse spin related sum rules (not all discussed):

- Transverse spin decomposition
- Burkhardt-Cottingham sum rule (& old Burkardt sum rule)
- ELT sum rule
- (new) Burkardt sum rule
- Schäfer-Teryaev sum rule
- BLT sum rule

I will discuss x and kt-integrations, comment on small-x and polarization, possibility of nodes, scale dependence, etc

Transverse spin decomposition

Transverse spin decomposition

Philip G. Ratcliffe, at spin98 in Protvino, hep-ph/9811348:

1.3 Global Sum Rules

Another important and intuitive decomposition is that of the z-axis projection of the total nucleon spin:

$$J_z^p = \frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta g + L_z^{q+g}, \tag{2}$$

together with the twin sum rule for the transverse projection:

$$J_T^p = \frac{1}{2} = \frac{1}{2}\Delta_T\Sigma + \Delta_T g + L_T^{q+g}. \tag{3}$$

I include the transverse-spin sum rule merely as a reminder of its existence. There are extra subtleties here: for example, the densities, $\Delta_T\Sigma$, have twist-3 contributions (absent for longitudinal polarisation).

Also Harindranath, Mukherjee, Ratabole, 2000; Hatta, Tanaka, Yoshida, 2013

What do we learn from the transverse spin decomposition?
In lectures on spin physics I \`dismissed' this transverse spin sum rule as having no new content because of the BC sum rule, but is that really true?

Burkhardt-Cottingham sum rule

One photon exchange hadronic tensor

$$W_A^{\mu\nu} = \frac{i\epsilon^{\mu\nu\rho\sigma} q_\rho}{P\cdot q}\left[S_\sigma g_1(x_B, Q^2) + \left(S_\sigma - \frac{S\cdot q}{P\cdot q}P_\sigma\right) g_2(x_B, Q^2)\right]$$

$$\int_0^1 dx\; g_2(x, Q^2) = 0$$

Burkhardt & Cottingham, 1970

This sum rule applies to the structure function, not the parton distribution

At tree level:
$$g_1(x_B, Q^2) = \frac{1}{2}\sum_{q,\bar{q}} e_q^2 g_1^q(x_B)$$

$$g_1(x_B, Q^2) + g_2(x_B, Q^2) = \frac{1}{2}\sum_{q,\bar{q}} e_q^2 g_T^q(x_B)$$

The BC sum rule applies at high Q, any power corrections are beyond twist-4

Other assumptions made by B&C were: rotational invariance in the proton rest frame, parity invariance, analyticity arguments (related to small x limit)

Burkhardt-Cottingham sum rule

The BC sum rule on the parton level can be translated into:

$$\int_{-1}^{1} dx\ g_2^q(x) = \int_0^1 dx\ \left[g_2^q(x) + g_2^{\bar{q}}(x)\right] = 0$$

This also happens to be the combination (q + q-bar) that is accessed in NC DIS

The BC sum rule for the structure function could also be satisfied by cancellation among quark flavors, but the above sum rule for parton distributions can be derived at the operator level for each flavor separately, from Lorentz invariance

The BC sum rule for the structure function can not be derived within the OPE though, as opposed to the other even and odd moments (NC, ± = $W^- \pm W^+$)

$$\int_0^1 dx x^n g_2^{NC,+}(x,Q^2) \;=\; \sum_q \frac{((g_V^q)^2 + (g_A^q)^2) n (d_n^{+q} - a_n^{+q})}{4(n+1)}\ , \quad n = 2,4\ \ldots$$

$$\int_0^1 dx x^n g_2^{-}(x,Q^2) \;=\; \sum_q \frac{((g_V^q)^2 + (g_A^q)^2) n (d_n^{-q} - a_n^{-q})}{4(n+1)}, \quad n = 1,3\ \ldots$$

E.g. Blümlein & Kochelev, 1996

Partonic version of the Burkhardt-Cottingham sum rule

The following derivation does use properties of a particular local operator:

Burkardt, 1995

$$\begin{aligned}\frac{1}{2}\mathrm{Tr}\left[\Phi(x)\gamma^\mu\gamma_5\right] &= \int\frac{d\lambda}{4\pi}e^{i\lambda x}\langle P,S|\overline{\psi}(0)\gamma^\mu\gamma_5\psi(\lambda n_-)|P,S\rangle \\ &= \lambda n_+^\mu g_1(x) + \frac{M}{P^+}S_T^\mu g_T(x) + \frac{\lambda M^2}{(P^+)^2}n_-^\mu g_3(x).\end{aligned} \tag{3.53}$$

For completeness we have included the twist-four distribution function g_3. If we take the first moment this leads to:

$$\frac{1}{2}\langle P,S|\overline{\psi}(0)\gamma^\mu\gamma_5\psi(0)|P,S\rangle = \int_{-1}^{1}dx\left[\lambda n_+^\mu g_1(x) + \frac{MS_T^\mu}{P^+}g_T(x) + \frac{\lambda M^2 n_-^\mu}{(P^+)^2}g_3(x)\right]. \tag{3.54}$$

Since the l.h.s. must be proportional to S^μ, one can easily calculate the proportionality factor, since we know that $P\cdot S = 0$. This leads to:

$$\int_{-1}^{1}dx\, g_1(x) = \int_{-1}^{1}dx\, g_T(x), \tag{3.55}$$

$$\int_{-1}^{1}dx\, g_1(x) = -2\int_{-1}^{1}dx\, g_3(x), \tag{3.56}$$

Hence:

$$\int_{-1}^{1}dx\ g_2^q(x) = \int_0^1 dx\ \left[g_2^q(x) + g_2^{\bar{q}}(x)\right] = 0$$

Partonic Burkhardt-Cottingham sum rule

$$\int_{-1}^{1} dx\ g_2^q(x, Q^2) = \int_0^1 dx\ \left[g_2^q(x,Q^2) + g_2^{\bar{q}}(x,Q^2)\right] = 0$$

This sum rule is stable under perturbative corrections & scale changes

despite some initial doubt (Mertig & Van Neerven, 1993)

Altarelli, Lampe, Nason & Ridolfi, 1994
Kodaira, Matsuda, Uematsu & Sasaki, 1994
Harindranath & Zhang, 1997
Belitsky, Ji, Lu & Osborne, 2001

Kodaira, Matsuda, Uematsu & Sasaki, 1994: “we expect that future experiments on g_2 will confirm the BC sum rule in its original form”

This overlooks two important issues: experiment cannot reach x=0 and at low x one is not guaranteed that the formalism applies in the first place (later more)

Chiral-odd version of partonic BC sum rule (old Burkardt sum rule)

$$\int_{-1}^{1} dx\, h_1(x) = \int_{-1}^{1} dx\, h_L(x) \ \Rightarrow\ \int_{-1}^{1} dx\, h_2(x) = 0$$

Burkardt, 1993 & 1995

Half the partonic BC sum rule

$$\int_{-1}^{1} dx\ g_2^q(x) = 0 \quad \not\Rightarrow \quad \int_{0}^{1} dx\ g_2^q(x) = 0$$

Using the e.o.m. and Lorentz invariance relations (and interchange of integrations) it follows:

$$\int_{0}^{1} dx\ g_2^q(x) = -g_{1T}^{(1)}(0) = \frac{m}{M} h_1(0)$$

D.B., 1996

Lorentz invariance relations (Bukhvostov, Kuraev, Lipatov, 1984) in its x-unintegrated form have been questioned on the basis of n-dependence of the fully unintegrated quark correlator introduced by its gauge link

Goeke, Metz, Pobylitsa & Polyakov, 2003

Albeit that deviations may be small

Metz, Schweitzer & Teckentrup, 2009

However, the fully unintegrated quark correlator *including* gauge link is not well-determined, so it is not clear whether the objection is valid

In any case, integral need not vanish and its contribution can cancel among q & q-bar

Half the partonic BC sum rule

$$\int_0^1 dx\ g_2^q(x)$$

If this vanishes there has to be a node in g_2^q (models for the structure function g_2 typically show a node, but are not clear on the individual quark contributions)

Stratmann 1993; Song, 1996; Weigel, Gamberg, 2000; Wakamatsu, 2000; ...

Can this be tested in experiment?

NC DIS (e.g. E155) probes q + q-bar, hence one needs CC DIS at an EIC

But again there are the objections: experiment cannot reach x=0 and at low x one is not guaranteed that the formalism applies

Possibility of $\delta(x)$ contributions has been considered

Heimann, 1973; Burkardt, 1995

These could either invalidate the BC sum rule or cancel any violation that is not associated with x=0, undermining the experimental check

x=0 contributions

$$g_2(x,Q^2) = g_2^{\text{observable}}(x,Q^2) + c\delta(x). \tag{4.33}$$

Then since experimenters cannot reach $x = 0$, the BC sum rule reads

$$\int_0^1 dx g_2^{\text{observable}}(x,Q^2) = -\frac{1}{2}c, \tag{4.34}$$

which is useless.

This pathology — a δ–function at $x = 0$ — is not as arbitrary as it looks. Instead it is an example of a disease known as a "$J = 0$ *fixed pole with non-polynomial residue*". First studied in Regge theory,[53,27] a $\delta(x)$ in $g_2(x,Q^2)$ corresponds to a *real constant term in a spin flip Compton amplitude which persists to high energy*. There is no fundamental reason to

Jaffe, 1996

A model calculation by Burkardt and Koike, 2002, does not exhibit $\delta(x)$ in g_T, but it does in the chiral-odd twist-3 distribution functions h_L and e

Due to possibility of $\delta(x)$ no node is needed (for the integral from -1 to 1 no node is needed in the first place since it can be an odd function of x simply)

small x contributions

Multi-regge pole cuts may lead to a very singular g_2 as $x \to 0$, which may invalidate the BC sum rule (for the structure function)

Heimann, 1973

Cf. also Anselmino, Efremov, Leader, 1995

Diffraction gives steep rise and invalidates BC sum rule

Ivanov, Nikolaev, Pronyaev, W. Schäfer, 1999

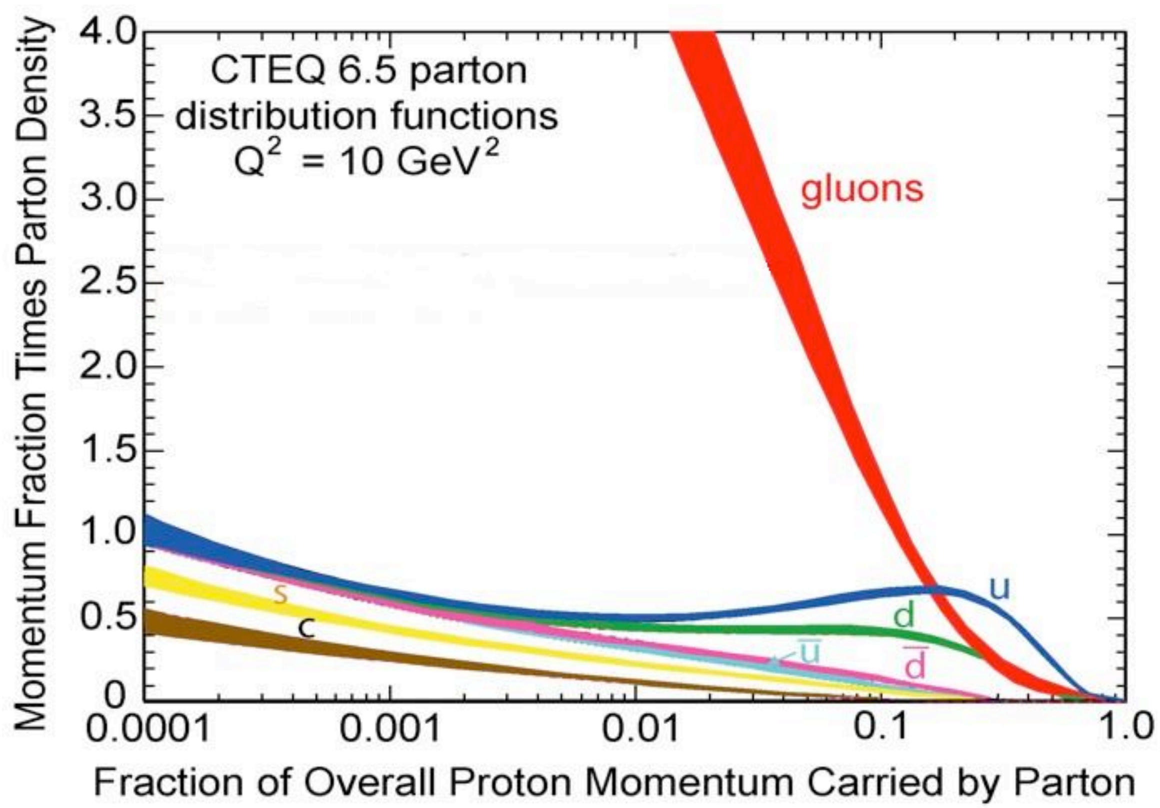
In a partonic picture small x arguments may invalidate the formalism of leading twist parton distributions, one may not be able to restrict to leading twist simply

Gluon diffusion towards small transverse momentum in ladder graphs in DGLAP treatment imply large nonperturbative contributions

A.H. Mueller, 1997

But see also, Ciafaloni, Colferai, Salam, 2000

Nonlinear effects



This is mainly an affair of gluons, but eventually it feeds into the quark distributions

When x decreases, the density of gluons (n) increases

At some point n becomes so large ($n \rightarrow O(1/\alpha_s)$) that the probability for gluons to interact approaches 1 ($n \times \sigma_{gg} \rightarrow 1$)

Scattering off a proton becomes scatter off multiple gluons simultaneously

It leads to nonlinear evolution equations, which show asymptotic solutions exhibiting saturation

Weizsäcker-Williams field

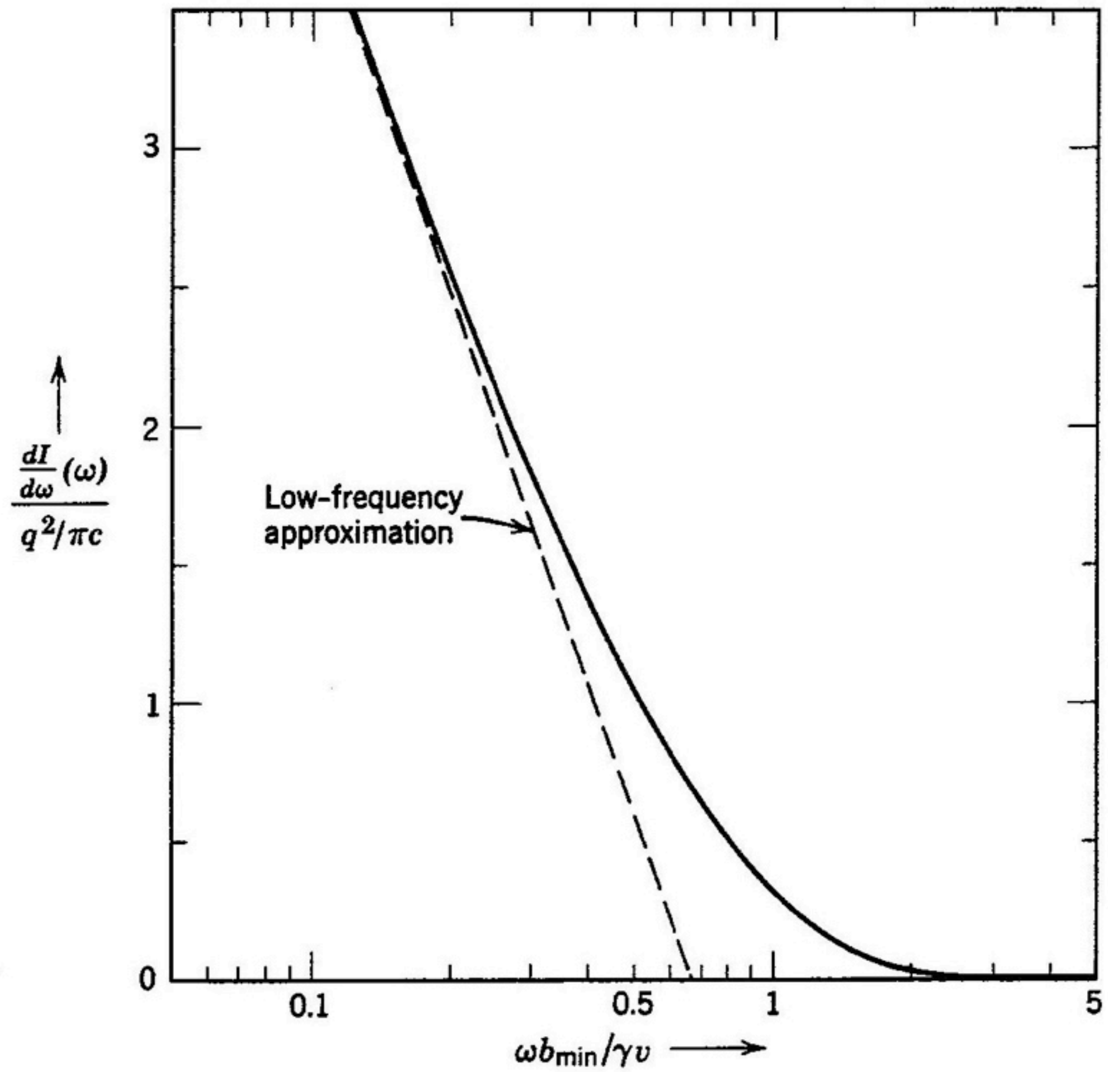
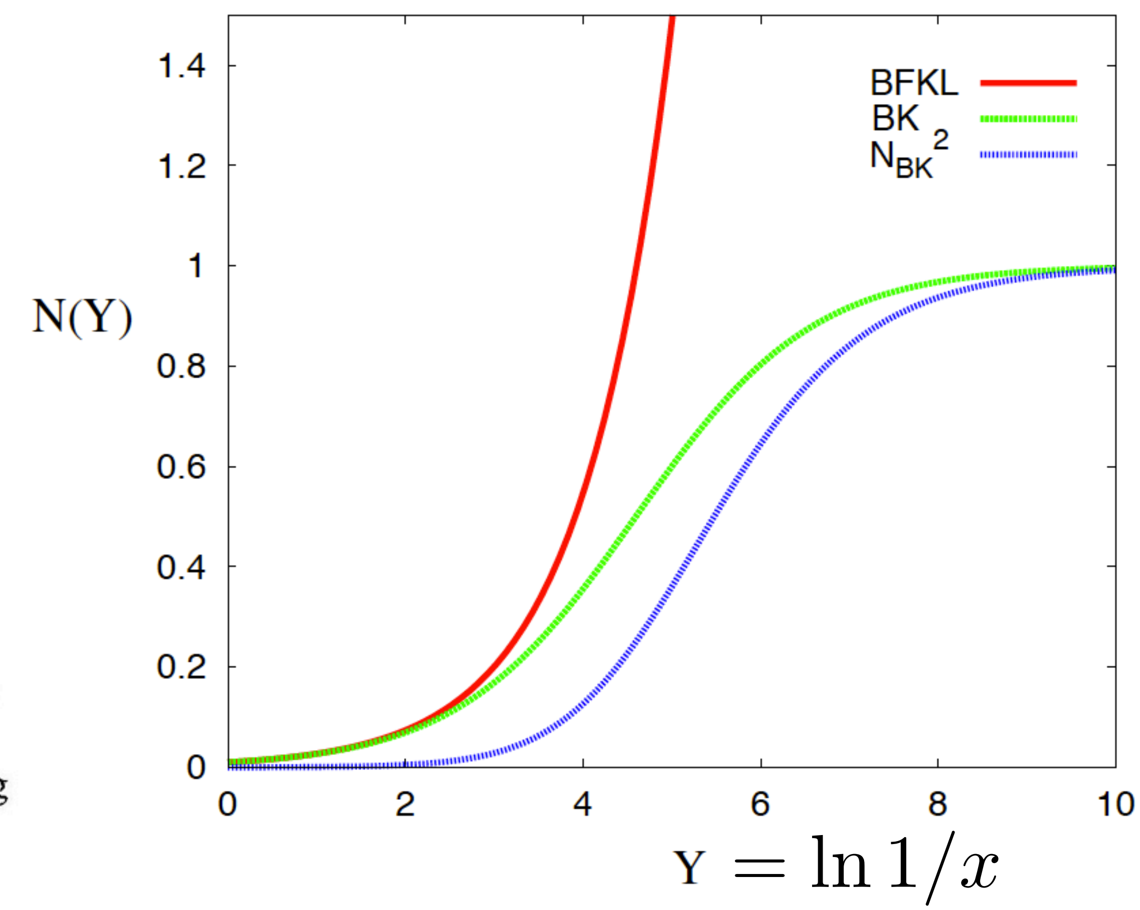


Figure 15.8 Frequency spectrum of virtual quanta for a relativistic particle, with the energy per unit frequency $dI(\omega)/d\omega$ in units of $q^2/\pi c$ and the frequency in units of $\gamma v/b_{\text{min}}$. The number of virtual quanta per unit energy interval is obtained by dividing by $\hbar^2\omega$.

Photon spectrum of a relativistic charge consists mostly of low energy photons

Dalitz & Yennie, 1957



Analogously for $g(x,Q^2)$: non-Abelian WW field consists of small-x gluons mainly

The WW gluon density exhibits saturation for x → 0 unlike the photon density

McLerran, Venugopalan, 1994

Nonlinear evolution equations

The first nonlinear evolution equation considered was the GLR equation:

$$\frac{\partial^2 xg(x,Q^2)}{\partial \ln 1/x \partial \ln Q^2} = \frac{\alpha_s N_c}{\pi} xg(x,Q^2) - \frac{\alpha_s^2 N_c}{R^2 Q^2}[xg(x,Q^2)]^2$$

Gribov, Levin & Ryskin, 1983; Laenen, Levin, 1995

But instead of looking at the gluon number density $g(x, Q^2) \propto \langle A^\dagger A \rangle$, at small x it becomes necessary to look at more general quantities than $\langle A^\dagger A \rangle^2$ ($\neq \langle A^\dagger A^\dagger A\, A \rangle$) such as the multiple gluon correlation function

$$N(x,r) = \langle V^\dagger(r) V(0) \rangle \qquad \text{with} \quad V = P \exp\left(i g_s \int ds^- A^+(r_\perp, s^-) \right) - 1$$

Typical of potential scattering

At low gluon density (small coupling g_s)

$$N(x,r) \to r^2 g(x, Q^2 = 1/r^2)$$

Nonlinear evolution equations

At small x: N(x,r) satisfies the nonlinear Balitsky-Kovchegov (BK) equation

$$\partial_Y \mathcal{N} = \chi(-\partial_L)\mathcal{N} - \mathcal{N}^2$$

Balitsky 1996; Kovchegov, 1999

$$\mathcal{N}(x,k) \equiv \int d^2 r e^{ikr} \frac{N(x,r)}{r^2} \qquad Y = \frac{\alpha_s N_c}{\pi} \ln\frac{1}{x} \qquad L = \ln k^2$$

BFKL kernel:

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)$$

$$\psi(z) = d \ln \Gamma(z)/dz$$

The BK equation reduces to the BFKL equation at low density (larger x, small r) and to the GLR equation at large x (in DLLA)

The BK equation exhibits saturation, as does the generalization to multiple Wilson line correlators $\langle V^\dagger \cdots V^\dagger V \cdots V \rangle$, that yields an infinite tower of coupled nonlinear evolution equations: JIMWLK equations

Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner 1997-2001

Gluon polarization at small x

Is there a BK equation for the polarized gluons? Probably not

Small-x effects are suppressed in the polarized case, i.e. Δg at small x is suppressed w.r.t. g

Evolution kernel does not have 1/x behavior, see e.g. Maul's CCFM study, 2002

$$\Delta P_{gg}(z) = \frac{2C_A(2-z)}{1-z}$$

Does $\Delta g(x)$ make sense when $g(x)$ does not anymore?

If $\Delta g(x) \approx 0$ in region around x_{min} from experiment, then that is probably no problem for the spin decomposition

But it is something to worry about

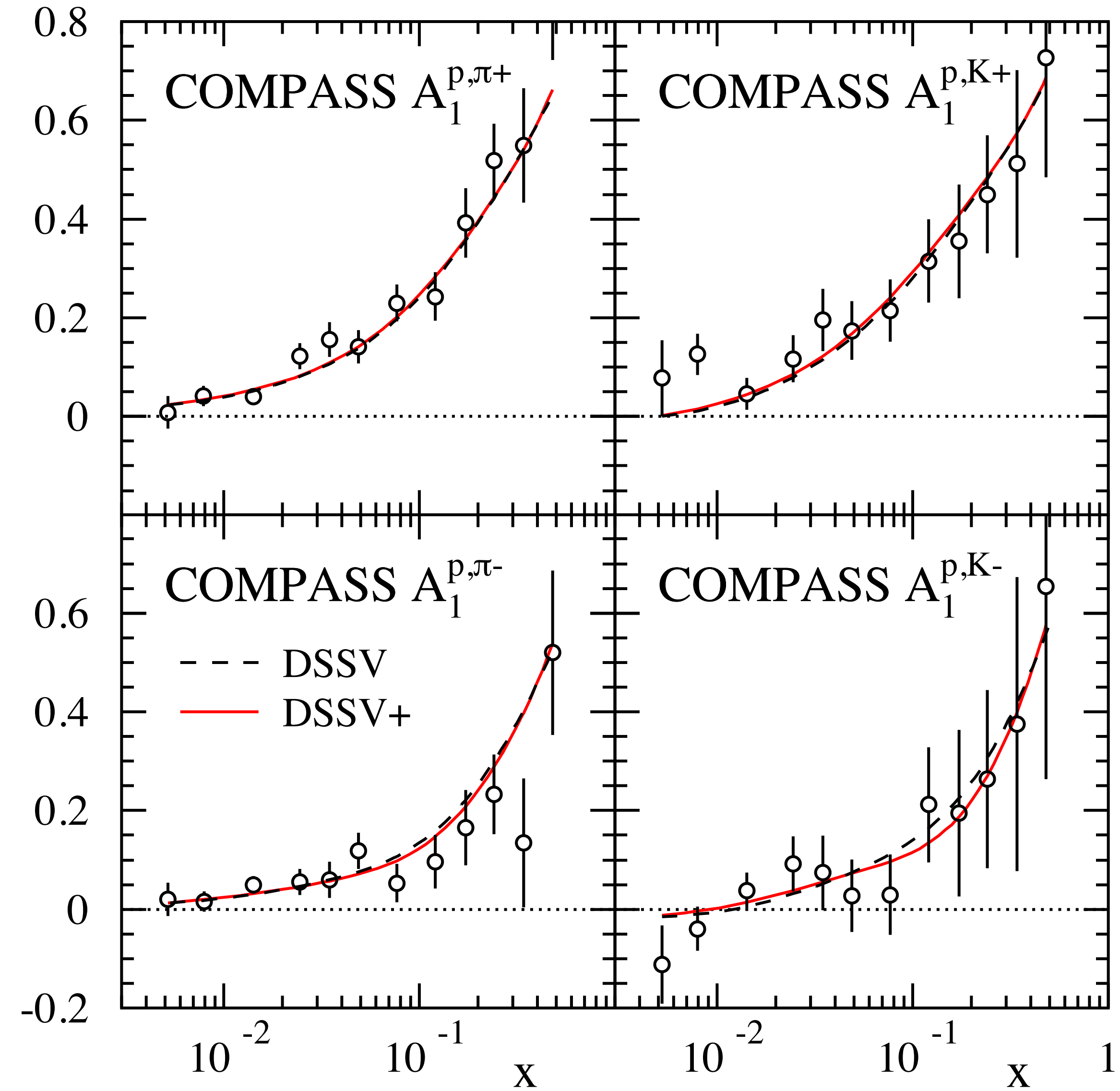
The Altarelli question

DSSV first moments at Q^2=10 GeV^2

	$x_{\min} = 0$	$x_{\min} = 0.001$	
	best fit	$\Delta\chi^2 = 1$	$\Delta\chi^2/\chi^2 = 2\%$
$\Delta u + \Delta\bar{u}$	0.813	$0.793\ ^{+0.011}_{-0.012}$	$0.793\ ^{+0.028}_{-0.034}$
$\Delta d + \Delta\bar{d}$	-0.458	$-0.416\ ^{+0.011}_{-0.009}$	$-0.416\ ^{+0.035}_{-0.025}$
$\Delta\bar{u}$	0.036	$0.028\ ^{+0.021}_{-0.020}$	$0.028\ ^{+0.059}_{-0.059}$
$\Delta\bar{d}$	-0.115	$-0.089\ ^{+0.029}_{-0.029}$	$-0.089\ ^{+0.090}_{-0.080}$
$\Delta\bar{s}$	-0.057	$-0.006\ ^{+0.010}_{-0.012}$	$-0.006\ ^{+0.028}_{-0.031}$
Δg	-0.084	$0.013\ ^{+0.106}_{-0.120}$	$0.013\ ^{+0.702}_{-0.314}$
$\Delta\Sigma$	0.242	$0.366\ ^{+0.015}_{-0.018}$	$0.366\ ^{+0.042}_{-0.062}$

de Florian, Sassot, Stratmann, Vogelsang, 2008

DIS asymmetry A_1



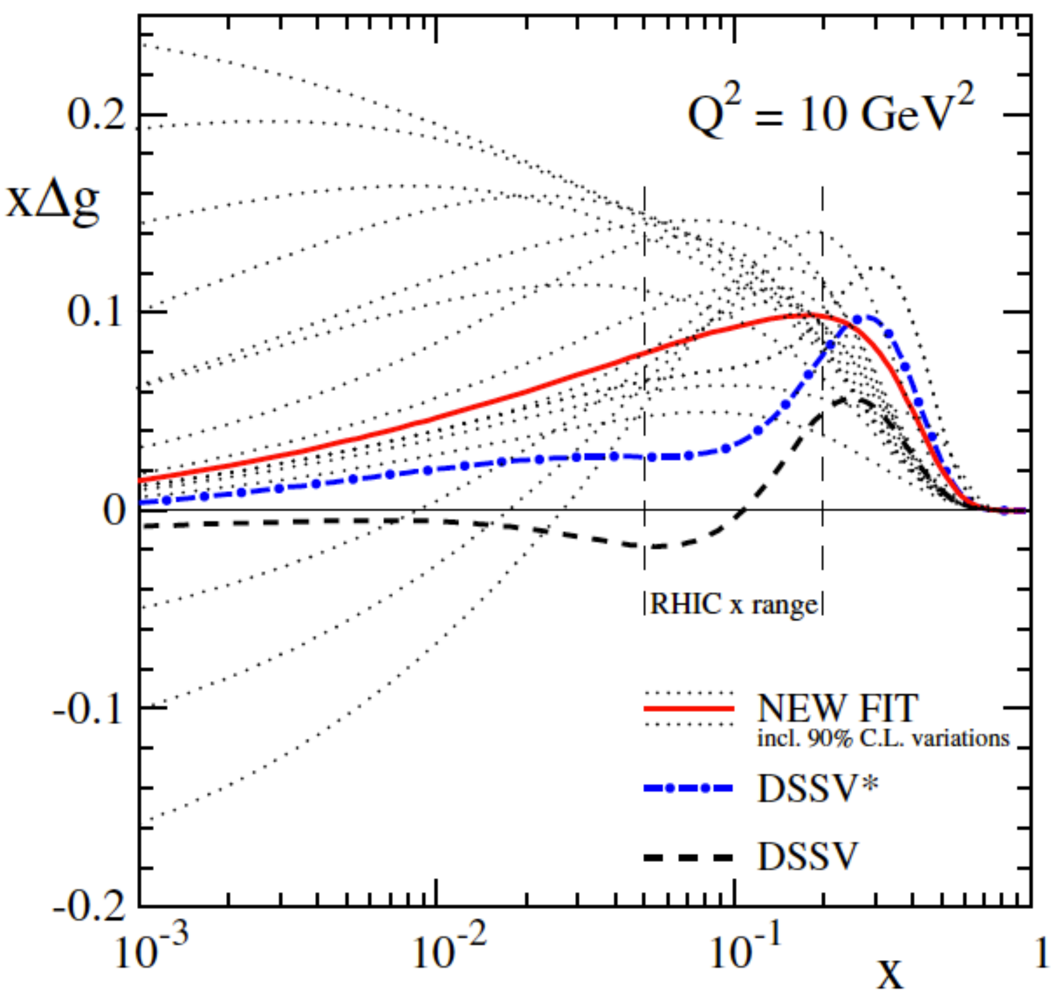
Altarelli questioned at some conference the DSSV fit because it suggests that there is a lot happening below $x = 10^{-3}$, where the measured DIS asymmetry is essentially zero

But in the NNPDF study and new DSSV Δg this is already much less the case

Δf	$\langle \Delta f(Q^2) \rangle^{[0,1]}$ NNDPFpol1.0	NNPDFpol1.1	$\langle \Delta f(Q^2) \rangle^{[10^{-3},1]}$ NNDPFpol1.0	NNPDFpol1.1	DSSV08
Δu^+	$+0.77 \pm 0.10$	$+0.79 \pm 0.07$	$+0.76 \pm 0.06$	$+0.76 \pm 0.04$	$+0.793^{+0.028}_{-0.034}$ $(+0.020)$
Δd^+	-0.46 ± 0.10	-0.47 ± 0.07	-0.41 ± 0.06	-0.41 ± 0.04	$-0.416^{+0.035}_{-0.025}$ (-0.042)
$\Delta \bar{u}$	—	$+0.06 \pm 0.06$	—	$+0.04 \pm 0.05$	$+0.028^{+0.059}_{-0.059}$ $(+0.008)$
$\Delta \bar{d}$	—	-0.11 ± 0.06	—	-0.09 ± 0.05	$-0.089^{+0.090}_{-0.080}$ (-0.026)
Δs	-0.07 ± 0.06	-0.07 ± 0.05	-0.06 ± 0.04	-0.05 ± 0.04	$-0.006^{+0.028}_{-0.031}$ (-0.051)
$\Delta \Sigma$	$+0.16 \pm 0.30$	$+0.18 \pm 0.21$	$+0.23 \pm 0.15$	$+0.25 \pm 0.10$	$+0.366^{+0.042}_{-0.062}$ $(+0.124)$

Table 12: Full and truncated first moments of the polarized quark distributions, Eq. (16), at $Q^2 = 10$ GeV^2, for NNPDFpol1.1, NNPDFpol1.0 (when available) and DSSV08. The uncertainties shown are one-sigma for NNPDF and Lagrange multiplier with $\Delta\chi^2/\chi^2 = 2\%$ for DSSV. The number in parenthesis for DSSV08 is the contribution that should be added to the truncated moment in order to obtain the full moment.

NNPDF, Nocera *et al.*, 2014



Errors on Δg from NNPDF large

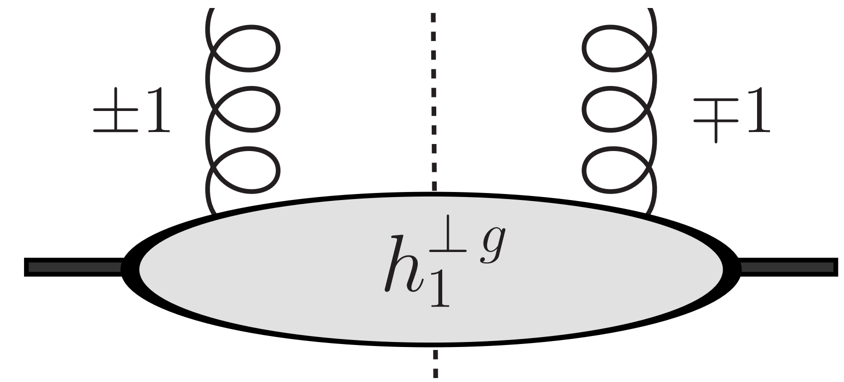
From RHIC asymmetry A_{LL}^{jet}

de Florian, Sassot, Stratmann, Vogelsang, 2014

New central fit yields truncated moment from 0.001 to 1 accounts for more than 90% of full moment at $Q^2 = 10$ GeV^2

Gluon polarization inside unpolarized protons

Is polarization completely irrelevant at small x then?

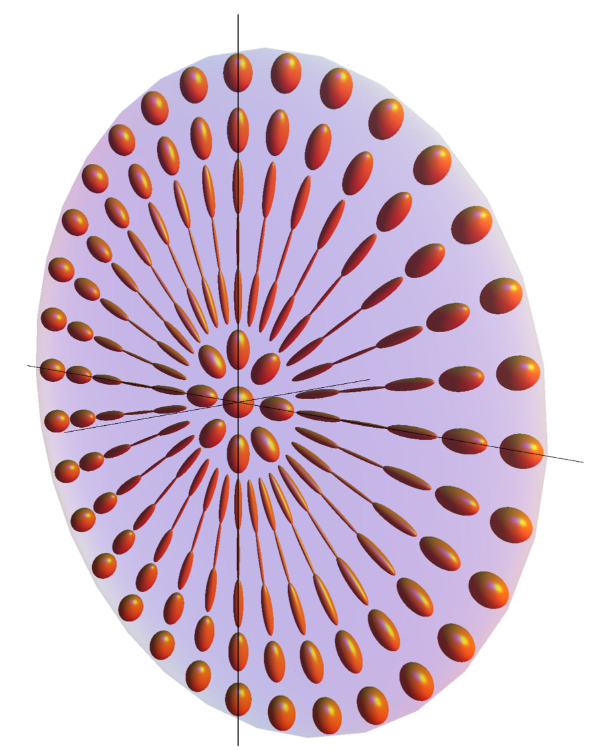


Δg corresponds to circularly polarized gluons

Linearly polarized gluons exist in unpolarized hadrons

Mulders, Rodrigues, 2001

an interference between
±1 helicity gluon states



For $h_1^{\perp\, g} > 0$ gluons prefer to be polarized along k_T,
with a $cos\ 2\phi$ distribution of linear polarization
around it, where $\phi = \angle(k_T, \varepsilon_T)$

Linearly polarization does grow with 1/x

It affects the transverse momentum distribution in $pp \to HX$ (Higgs production)

Catani & Grazzini, 2010; Sun, Xiao, Yuan, 2011; D.B., Den Dunnen, Pisano, Schlegel, Vogelsang, 2012

What do we know about the polarization?

At small x the WW (or CGC) gluon field and the dipole distribution have been studied:

$$h_{1,WW}^{\perp g} \ll f_{1,WW}^{\perp g} \quad \text{for } k_\perp \ll Q_s, \qquad h_{1,WW}^{\perp g} = 2 f_{1,WW}^{\perp g} \quad \text{for } k_\perp \gg Q_s$$

$$x h_{1,DP}^{\perp g}(x, k_\perp) = 2 x f_{1,DP}^{g}(x, k_\perp)$$

Metz, Zhou, 2011

At small x the k_T-factorization approach implies maximum polarization too:

$$\Phi_g^{\mu\nu}(x, \boldsymbol{p}_T)_{\text{max pol}} = \frac{2}{x} \frac{p_T^\mu p_T^\nu}{\boldsymbol{p}_T^2} f_1^g$$

Catani, Ciafaloni, Hautmann, 1991

One can also consider the perturbative tail, which is calculable

$$\tilde{f}_{g/P}(x, b^2; \mu, \zeta) = \sum_{i=g,q} \int_x^1 \frac{d\hat{x}}{\hat{x}} C_{i/g}(x/\hat{x}, b^2; g(\mu), \mu, \zeta) f_{i/P}(\hat{x}; \mu) + \mathcal{O}((\Lambda_{\text{QCD}} b)^a)$$

What do we know about the polarization?

Fourier transform:

$$\tilde{h}_1^{\perp\, g}(x,b^2) = \int d^2\boldsymbol{p}_T\, \frac{(\boldsymbol{b}\cdot\boldsymbol{p}_T)^2 - \frac{1}{2}\boldsymbol{b}^2\boldsymbol{p}_T^2}{b^2 M^2}\, e^{-i\boldsymbol{b}\cdot\boldsymbol{p}_T}\, h_1^{\perp g}(x,p_T^2)$$

$$= -\pi \int dp_T^2 \frac{p_T^2}{2M^2} J_2(bp_T) h_1^{\perp g}(x,p_T^2)$$

The perturbative tail is driven by the unpolarized gluon distribution:

$$\tilde{f}_1^g(x,b^2;\mu_b^2,\mu_b) = f_{g/P}(x;\mu_b) + \mathcal{O}(\alpha_s)$$

$$\tilde{h}_1^{\perp g}(x,b^2;\mu_b^2,\mu_b) = \frac{\alpha_s(\mu_b)C_A}{2\pi}\int_x^1 \frac{d\hat{x}}{\hat{x}}\left(\frac{\hat{x}}{x}-1\right) f_{g/P}(\hat{x};\mu_b) + \mathcal{O}(\alpha_s^2)$$

Nadolsky, Balazs, Berger, Yuan, 2007; Catani, Grazzini, 2010

There is no theoretical reason why the distribution should be small, especially at small x, except for its significant suppression by α_s

small x contributions

The transverse spin decomposition as discussed by Ratcliffe (1998) and Harindranath, Mukherjee, Ratabole (2000) deals with the twist-3 gluon distribution $\Delta_T g$ or g_T^g

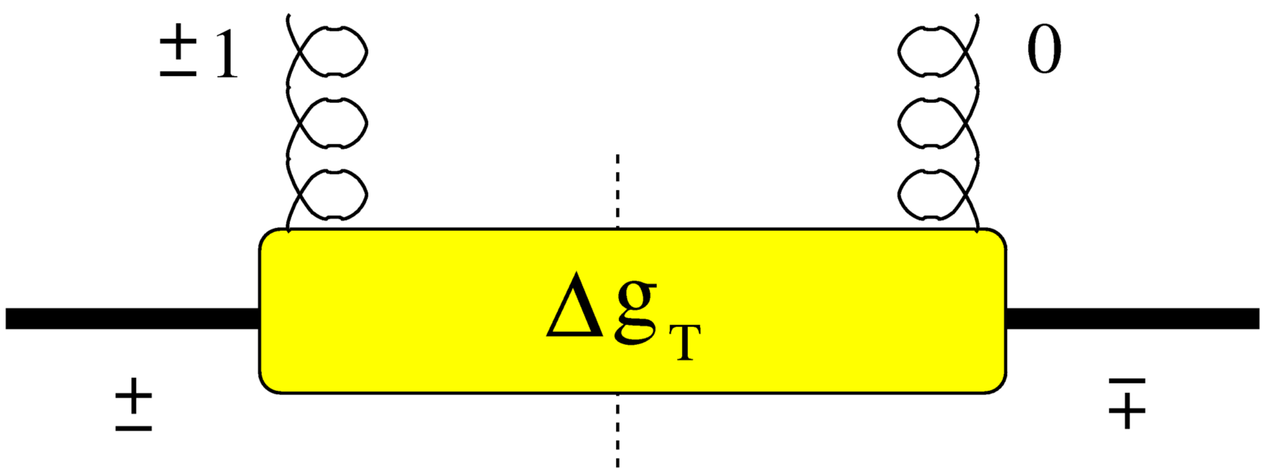
We find (combining the polarizations)

$$\Gamma^{ij}(x) = \frac{x}{2}\frac{P^+}{M}[-g_T^{ij}G(x) - S_L i\epsilon_T^{ij}\Delta G(x)]$$

$$\Gamma^{i-}(x) = \frac{x}{2} i\epsilon_T^{S_T i}\Delta G_{3T}(x),$$

$$\Gamma^{ij,l}(x) = \frac{x}{2} i\epsilon_T^{ij}S_T^l\Delta H_{3T}(x),$$

where $G(x)=\int d^2\boldsymbol{k}_T G(x,\boldsymbol{k}_T^2)$ and similarly for ΔG_{3T} and ΔH_{3T}, while $\Delta G(x)=\int d^2\boldsymbol{k}_T\Delta G_L(x,\boldsymbol{k}_T^2)$. The functions ΔG_{3T} and ΔH_{3T} are in essence the functions H_1 and H_2 of Ref. [6].



Ji, 1992
Ali, Hoodbhoy, 1993
Mulders, Rodrigues, 2001

Probably small-x effects are not dominant for $\Delta_T g$, like for Δg (cf. however Jian Zhou's talk)

If there is no $\delta(x)$ contribution and the region below x_{min} from experiment is negligible, then the transverse spin sum rule can be considered testable

Transverse spin decomposition

$$\Delta\Sigma = \int_0^1 dx\,\left[g_1^q(x) + g_1^{\bar{q}}(x)\right] = \int_{-1}^1 dx\; g_1^q(x) = \int_{-1}^1 g_T^q(x) = \Delta_T\Sigma$$

$$\Delta g = \int_0^1 dx\; g(x) = \int_0^1 dx\,\left[g^+ - g^-\right] = \int_0^1 dx\; g_T^g(x) = \Delta_T g$$

Hatta, Tanaka, Yoshida, 2012

$$\Delta\Sigma = \Delta_T\Sigma \quad \& \quad \Delta g = \Delta_T g \quad \Rightarrow \quad L_z^{q+g} = L_T^{q+g}$$

In this sense the transverse spin decomposition does indeed not add any new content

One may wish to check it experimentally though, it might shed light on the small x issue

Also there is a twist to the spin decomposition story: the OAM term cannot be split

The split of the OAM term does not need to be equal to the split of L_z
$L_z^{q,g}$ are not local operators (neither is the gluon spin term, but it does rotate nicely) and need not rotate trivially (i.e. difference to $L_T^{q,g}$ need not be zero due to Lorentz invariance)

See also: Ji, Xiong, Yuan, 2012; Leader, 2013; Harindranath, Kundu, Mukherjee, Ratabole, 2013

Transverse spin decomposition

$$\frac{\langle PS|W^i_{q-\mathrm{spin}}|PS\rangle}{2P^+(2\pi)^3\delta^3(0)} = \frac{1}{4}\frac{\langle PS|\int d^3x\bar{\psi}\gamma_5\gamma^i\psi|PS\rangle}{(2\pi)^3\delta^3(0)} = \frac{1}{2}\int dx g_T(x)S^i = \frac{1}{2}\Delta\Sigma S^i$$

$$\frac{\langle PS|W^i_{g-\mathrm{spin}}|PS\rangle}{2P^+(2\pi)^3\delta^3(0)} = S^i\int dx\,\mathcal{G}_{3T}(x) = S^i\Delta G$$

$$\int\frac{d\lambda}{2\pi}e^{i\lambda x}\langle PS|F^{+i}(0)WF^{+-}(\lambda n)|PS\rangle = -ix\mathcal{G}_{3T}(x)P^+\epsilon^{ij}S_j$$

$$\frac{\langle PS|W^i_{q,g}|PS\rangle}{2P^+(2\pi)^3\delta^3(0)} \equiv J_{q,g}S^i$$

Hatta, Tanaka, Yoshida, 2013

an additional frame-dependent contribution to $J_{q,g}$

$$J_{q,g} = \frac{1}{2}(A_{q,g}+B_{q,g}) + \frac{P^3}{2(P^0+M)}\bar{C}_{q,g}$$

See also discussion following Ji, Xiong, Yuan, 2012, by Leader, 2012 and Harindranath, Kundu, Mukherjee, Ratabole, 2013

Only the total orbital angular momentum of the full decomposition is rotationally invariant

Burkardt sum rule

Burkardt sum rule

Integral of Sivers ("rhymes with rivers") function over all of x and k_T satisfies

$$\sum_{a=q,g} \int f_{1T}^{\perp(1)a}(x)\, dx = 0$$

Burkardt, 2004

Involves the conventional transverse moment:

$$f_{1T}^{\perp(1)}(x) \equiv \int d^2k_T \frac{k_T^2}{2M^2} f_{1T}^{\perp}(x, k_T^2)$$

Without worrying about QCD corrections one has the (gauge invariant) relation:

$$f_{1T}^{\perp(1)}(x) = -\frac{g}{2M} T(x, S_T)$$

D.B., Mulders, Pijlman, 2003

$T(x,S_T)$ is the collinear twist-3 Qiu-Sterman function $T_F(x,x)$:

Qiu & Sterman,1991

$$T_F(x,x) \overset{A^+=0}{\propto} \mathrm{F.T.}\, \langle P|\, \overline{\psi}(0) \int d\eta^- \; F^{+\alpha}(\eta^-)\, \gamma^+ \psi(\xi^-)\, |P\rangle$$

Burkardt sum rule already (approximately) satisfied by up and down quarks which are approximately equal in magnitude and opposite in sign

TMDs and collinear pdfs

Large transverse momentum (perturbative) tail of TMD determined by collinear pdf

$$f_1(x, \boldsymbol{p}_T^2) \overset{\boldsymbol{p}_T^2 \gg M^2}{\sim} \alpha_s \, \frac{1}{\boldsymbol{p}_T^2} \, (K \otimes f_1)\,(x)$$

Tail of Sivers function determined by the collinear twist-3 Qiu-Sterman function

$$f_{1T}^{\perp}(x, \boldsymbol{p}_T^2) \overset{\boldsymbol{p}_T^2 \gg M^2}{\sim} \alpha_s \, \frac{M^2}{\boldsymbol{p}_T^4} \, (K' \otimes T_F)\,(x)$$

Ji, Qiu, Vogelsang, Yuan, 2006; Koike, Vogelsang, Yuan, 2008

One has to be careful when considering integrals over all transverse momenta
Convergence issue and does not automatically yield collinear pdfs

$$\int d\boldsymbol{k}_T f_1(x, \boldsymbol{k}_T; \mu, \zeta) \overset{?}{=} f_1(x; \mu) \qquad \zeta = 2M_p^2 x^2 e^{2(y_P - y_s)}$$

Collins, 2011

Bessel moments

To avoid the convergence issue one can consider Bessel moments:

$$\tilde{f}^{(n)}(x, \boldsymbol{b}_T^2) = n! \left(-\frac{2}{M^2} \partial_{\boldsymbol{b}_T^2} \right)^n \tilde{f}(x, \boldsymbol{b}_T^2)$$

Generalization of the conventional transverse moments

$$\tilde{f}^{(n)}(x, \boldsymbol{b}_T^2) = n! \left(-\frac{2}{M^2} \partial_{\boldsymbol{b}_T^2} \right)^n \tilde{f}(x, \boldsymbol{b}_T^2) \xrightarrow{\boldsymbol{b}_T^2 \to 0} f^{(n)}(x)$$

$$\tilde{f}_{1T}^{\perp(1)}(x, \boldsymbol{b}_T^2) \xrightarrow{\boldsymbol{b}_T^2 \to 0} f_{1T}^{\perp(1)}(x)$$

The limit should be considered with care

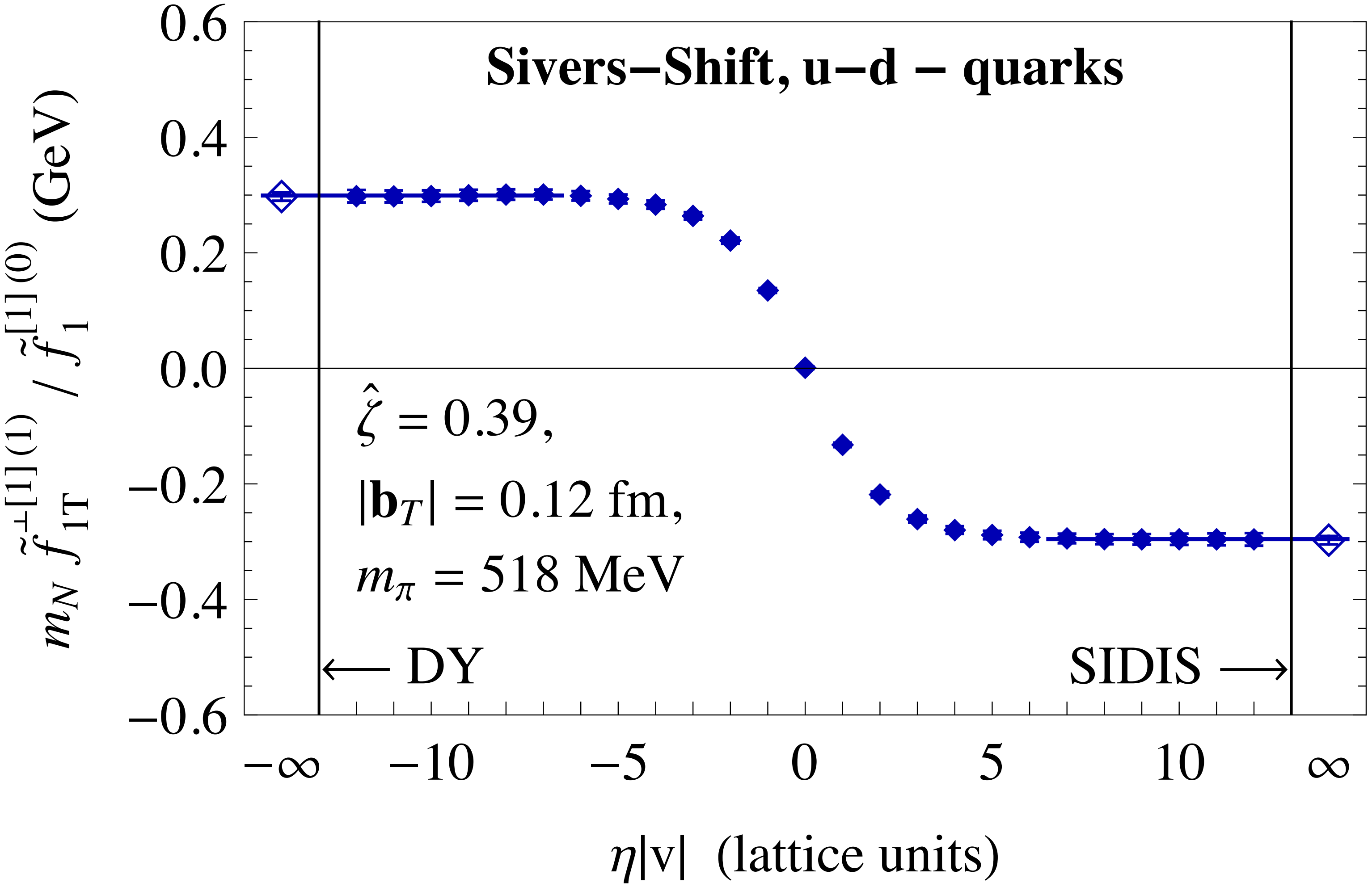
For finite b_T one can calculate ratios of Bessel moments that in principle can be evaluated on the lattice, e.g. the so-called Sivers shift:

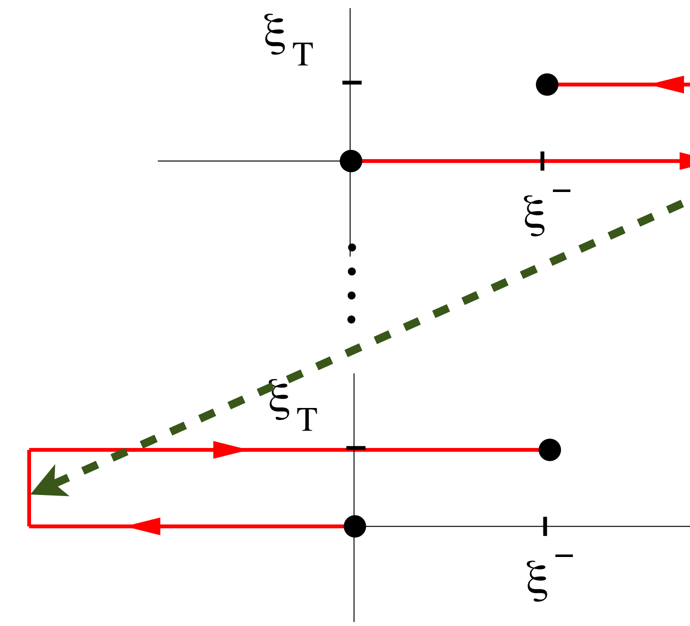
$$\begin{aligned}
\langle p_y(x) \rangle_{TU}^{\mathcal{B}_T} &= \left. \frac{\int d|p_T|\,|p_T| \int d\phi_p \frac{2J_1(|p_T|\mathcal{B}_T)}{\mathcal{B}_T} \, \sin(\phi_p - \phi_S)\, \Phi^{(+)[\gamma^+]}(x, p_T, P, S, \mu^2, \zeta)}{\int d|p_T|\,|p_T| \int d\phi_p J_0(|p_T|\mathcal{B}_T)\, \Phi^{(+)[\gamma^+]}(x, p_T, P, S, \mu^2, \zeta)} \right|_{|S_T|=1} \\
&= M \frac{\tilde{f}_{1T}^{\perp(1)}(x, \mathcal{B}_T; \mu^2, \zeta)}{\tilde{f}_1^{(0)}(x, \mathcal{B}_T; \mu^2, \zeta)}
\end{aligned}$$

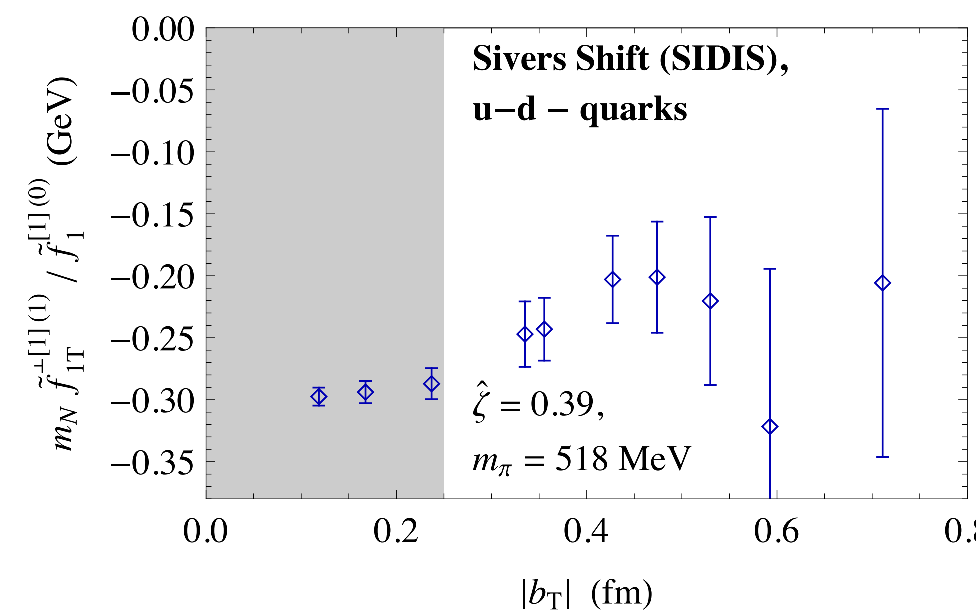
D.B., Gamberg, Musch, Prokudin, 2011

Sivers function on the lattice

Musch, Hägler, Engelhardt, Negele & Schäfer, 2012







The first `first-principle' demonstration in QCD that the Sivers function is nonzero
It clearly corroborates the sign change relation

$$f_{1T}^{\perp[\mathrm{SIDIS}]} = -f_{1T}^{\perp[\mathrm{DY}]}$$

compatible with fits and models:
up Sivers ($f_{1T}^{\perp}$) of SIDIS < 0 and down Sivers of SIDIS > 0

Qiu-Sterman function from the lattice

The limit $b_T \to 0$ tells us something about the Qiu-Sterman function

$$f_{1T}^{\perp(1)}(x) \equiv \int d^2k_T \, \frac{k_T^2}{2M^2} f_{1T}^{\perp}(x, k_T^2) \propto T_F(x,x)$$

$$\lim_{b_T \to 0} \tilde{f}_{1T}^{(1)[+]}(x, b_T^2; \mu, \zeta) \stackrel{?}{=} \frac{T_F(x,x;\mu)}{2M}$$

'?' because of rapidity dependence of r.h.s., identification meaningful when viewed as part of the full cross section expression, just like for:

$$\lim_{b_T \to 0} \tilde{f}_1^{(0)}(x, b_T^2; \mu, \zeta) \stackrel{?}{=} f_1(x;\mu)$$

Nevertheless, a very interesting limit to consider, since Qiu-Sterman function itself is intrinsically non-local along the lightcone and cannot be evaluated on the lattice

$$T_F(x,x) \overset{A^+=0}{\propto} \mathrm{F.T.} \, \langle P| \, \overline{\psi}(0) \int d\eta^- \; F^{+\alpha}(\eta^-) \gamma^+ \, \psi(\xi^-) \, |P\rangle$$

But first Bessel-moment of Sivers function *can* be evaluated (for given rapidity)

The limit $b_T \to 0$ of Sivers shift tells us about the Qiu-Sterman function

$$\lim_{b_T \to 0} \tilde{f}_{1T}^{(1)[+]}(x, b_T^2; \mu, \zeta) \stackrel{?}{=} \frac{T_F(x, x; \mu)}{2M}$$

This is especially promising if the limits $b_T \to 0$ and large ζ are constant/flat

$$\hat{\zeta} = \frac{\zeta}{2m_N} = \frac{\vec{v} \cdot \vec{P}}{\sqrt{|\vec{v}^{\,2}|}\sqrt{P^2}} = \sinh(y_P - y_v) \gg \frac{\Lambda_{\text{QCD}}}{2m_N} \approx 0.1$$

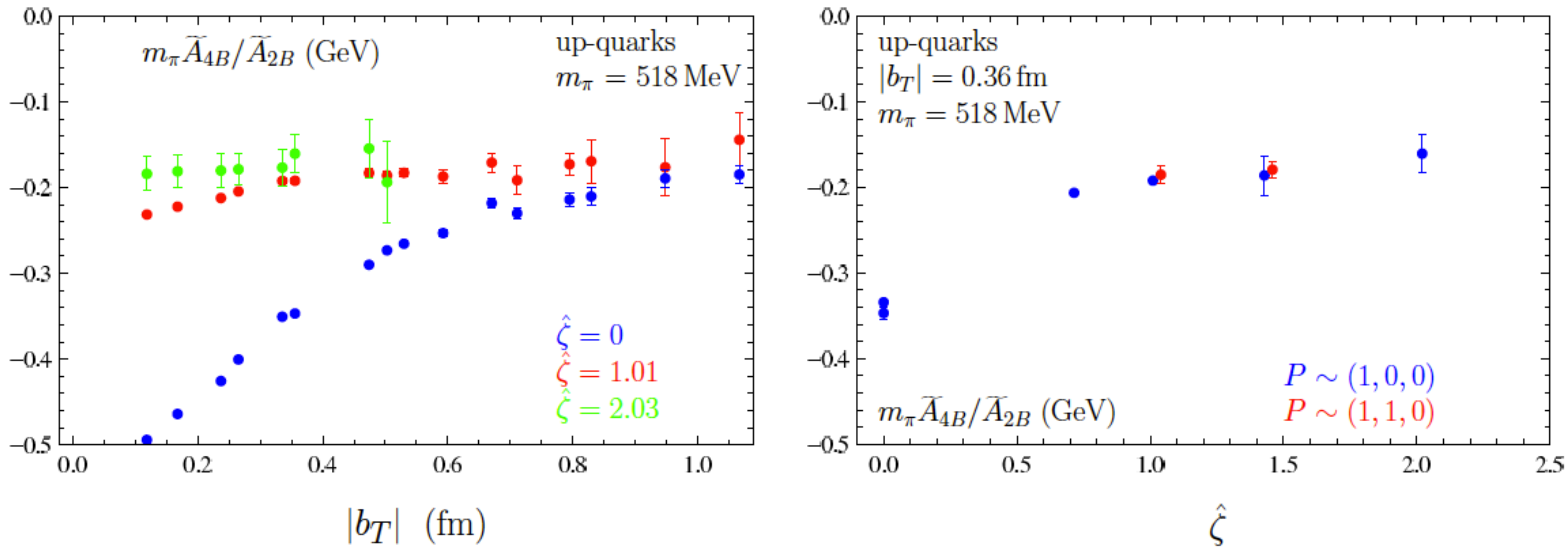


Figure 3: Generalized Boer-Mulders shift in the $\eta \to \infty$ SIDIS limit as a function of $|b_T|$ (left) and $\hat{\zeta}$ (right). In the left panel, the data in the region below $|b_T| \approx 0.25$ fm may be significantly affected by finite lattice cutoff effects. In the right panel, the congruence of the data obtained for P in different directions exhibits the good rotational properties of the calculation.

Engelhardt, 2013

Burkardt sum rule in terms of Qiu-Sterman function:

$$\sum_q \int_{-1}^{1} T^q(x, S_T)\, dx = -\int_0^1 T^g(x, S_T)\, dx$$

There are experimental indications (COMPASS deuteron SIDIS & RHIC A_N at midrapidity) and theoretical arguments (large Nc) that T^g is small

Combining such a valence scenario with lattice data on Sivers shift yields:

$$\int_{-1}^{1} T^u(x, S_T)\, dx \approx -\int_{-1}^{1} T^d(x, S_T)\, dx$$

$$\Longrightarrow \int_{-1}^{1} T^{u-d}(x, S_T)\, dx \approx 2\int_{-1}^{1} T^u(x, S_T)\, dx > 0$$

Taking the lattice evaluation literally indicates the integral over T^u is not small

It does not exclude the option of a node though, which has been suggested

x dependence of Qiu-Sterman function

$$T(x,S_T) = i\frac{M}{P^+}\int\frac{d\lambda}{2\pi}e^{i\lambda x}\langle P,S|\,\overline{\psi}(0)\Gamma_\alpha\int d\eta\; F^{+\alpha}(\eta n_-)\;\psi(\lambda n_-)\,|P,S\rangle$$

$$\Gamma_\alpha \equiv \frac{\epsilon_{T\beta\alpha}S_T^\beta\; \not{n}_-}{2iMP^+}$$

Qiu-Sterman function has been ``approximated” in two ways:

1) $$T(x,S_T) \xrightarrow{\int d\eta^- F^{+\alpha}(\eta^-)\to\langle F^{+\alpha}\rangle} f_1(x)$$ Qiu & Sterman, 1991

2) $$T(x,S_T) \xrightarrow{\int d\eta^- F^{+\alpha}(\eta^-)\to F^{+\alpha}(0^-)\int d\eta^-} \tilde{g}_T(x)$$ D.B., 2011

This second option yields in an unintegrated ESGM relation

Ehrnsperger, Schäfer, Greiner, Mankiewicz, 1994

Since $\tilde{g}_T(x)$ has a node, so does $T(x, S_T)$ *if* the unintegrated ESGM relation holds

This does not imply that the integral of $T(x,S_T)$ vanishes

x dependence of Qiu-Sterman function

The original integrated ESGM relation implies:

$$\sum_q e_q^2 \int_{-1}^{1} T^q(x, S_T)\, dx \propto d_2 \approx 0$$

Ehrnsperger, Schäfer, Greiner, Mankiewicz, 1994

$$d_2 = 3 \int_0^1 x^2 \; g_2(x)|_{\text{twist-3}}\, dx$$

E.g. Blümlein & Kochelev, 1996

The SLAC E155 experiment obtained $d_2 = 0.0032 \pm 0.0017$ for the proton (2003)

$\sum_q \int_{-1}^{1} T^q(x, S_T)\, dx$ and $\sum_q e_q^2 \int_{-1}^{1} T^q(x, S_T)\, dx$ cannot both be zero

except for vanishing integrals for u and d separately, but lattice disfavors this option

This may indicate that the ESGM relation may simply not be valid, but still it does not exclude the option of a node

Node in Sivers function

$$f_{1T}^{\perp q}(x, k_\perp^2) = -\mathcal{N}_q(x) h(k_\perp) f_{q/A}(x, k_\perp^2), \qquad (16)$$

Kang, Prokudin, 2012

where the extra $k_\perp$ dependence $h(k_\perp)$ is given by

$$h(k_\perp) = \sqrt{2e}\,\frac{M}{M_1} e^{-\mathbf{k}_\perp^2/M_1^2}, \qquad (17)$$

with M the nucleon mass, and M_1 a fitting parameter. The x-dependent part $\mathcal{N}_q(x)$ will be parametrized as

$$\mathcal{N}_q(x) = N_q x^{\alpha_q}(1-x)^{\beta_q} \frac{(\alpha_q+\beta_q)^{(\alpha_q+\beta_q)}}{\alpha_q^{\alpha_q}\beta_q^{\beta_q}}(1-\eta_q x).$$

SIDIS data fit yielded node in u Sivers, but not in d Sivers, but then Brahms data not describable when one assumes the Sivers-QS relation

It also assumes that A_N comes from QS effect, which need not be the case

Kanazawa, Koike, Metz, Pitonyak, 2014

Suggested solution of the sign mismatch problem allows for a node in $T(x,S_T)$

Node in Sivers function

Nodes can be *at different places for different flavors*, although one expects:

$$f_{1T}^{\perp u}(x, k_T^2) = -f_{1T}^{\perp d}(x, k_T^2) + \mathcal{O}(N_c^{-1})$$

Pobylitsa, 2003; Drago, 2005

Some model calculations show a node, but not for all flavors

d: Lu, Ma, 2004; Courtoy, Fratini, Scopetta, Vento, 2008

u: Bacchetta, Conti, Radici, 2008

However most model calculations of the Sivers function do not show a node

Brodsky, Hwang, Schmidt, 2002; Yuan, 2003; Bacchetta, Schäfer, Yang, 2004; Cherednikov, D'Alesio, Kochelev, Murgia, 2006; Gamberg, Goldstein, Schlegel, 2008; Courtoy, Scopetta, Vento, 2009; Pasquini, Yuan, 2010

These model calculations of the Sivers functions all consider the gauge link to lowest nontrivial order in g, the first order expansion of the Wilson line

It is unclear what the size of the higher order corrections is and whether these could change the sign in a particular x region

Overall sign relation test

Nodes are very important for the test of the Sivers sign relation

$$f_{1T}^{\perp[\text{SIDIS}]} = -f_{1T}^{\perp[\text{DY}]} \qquad \text{to be tested}$$

For the experimental test the functions must be compared at the same x and k_T^2

Possible future EIC data could be directly compared with Drell-Yan data,
but other SIDIS data from HERMES, COMPASS, JLab, require TMD evolution

The Sivers function may have a scale dependent node as a function x and/or k_T
D.B., 2011; Kang, Qiu, Vogelsang, Yuan, 2011

TMD evolution of the Sivers *asymmetry* is under control by now
Aybat, Prokudin & Rogers, 2012; Anselmino, Boglione, Melis, 2012; Sun & Yuan, 2013; Echevarria, Idilbi, Kang, Vitev, 2014; ...

But TMD evolution of the first transverse moment of Sivers *function* is very difficult
It is not autonomous, except for the nonsinglet part at large x

This is relevant for demonstrating the scale invariance of the Burkardt sum rule

Evolution of the QS function

Evolution of T_F is known; tricky because $dT_F(x,x)/d\ln\mu$ depends on $T_F(x,y)$ for $x \neq y$

Kang, Qiu, 2009; Zhou, Yuan, Liang, 2009; Vogelsang, Yuan, 2009; Braun, Manashov, Pirnay, 2009

The evolution does simplify, however, in the large x limit in which case the integration regions shrink to a point. One obtains

$$\mu \frac{d}{d\mu} \mathcal{T}_{q,F}(x,x) = \frac{\alpha_s}{\pi} \int_x^1 \frac{d\xi}{\xi} P_{q,F}^{NS,z\to 1}(z) \mathcal{T}_{q,F}(\xi,\xi) \,, \quad (47)$$

where, retaining singular terms at $z \to 1$ only

$$P_{q,F}^{NS,z\to 1}(z) = 2C_F \left[\frac{1}{(1-z)_+} + \frac{3}{4}\delta(1-z) \right] - N_c \delta(1-z) \,.$$

Simplification in the large x limit

Braun, Manashov, Pirnay, 2009

It evolves logarithmically with Q^2, but faster than f_1

$$\mathcal{T}_{q,F}(x,x;Q^2)/F_1(x,Q^2) \sim \left(\frac{\alpha_s(Q)}{\alpha_s(\mu_0)} \right)^{2N_c/b_0} \,,$$

$$g_2^{tw.-3}(x,Q^2)/F_1(x,Q^2) \sim \left(\frac{\alpha_s(Q)}{\alpha_s(\mu_0)} \right)^{N_c/b_0} \,.$$

Effect of large x evolution on Burkardt sum rule studied by Ratcliffe & Teryaev (2014)

Final comments on sum rules

Bessel moment of Sivers function is safer to consider than conventional transverse moment, but not clear whether sum rule applies for nonzero b_T

$$\sum_{a=q,g} \int \tilde{f}_{1T}^{\perp(1)[\pm]a}(x, b_T^2)\, dx \stackrel{?}{=} 0$$

Similar comments apply to the Schäfer-Teryaev sum rule (2000) for Collins function:

$$\sum_h \int dz\, z\, H_1^{\perp(1)}(z) = 0, \qquad H_1^{\perp(1)}(z) \equiv z^2 \int d^2\mathbf{k}_T \frac{\mathbf{k}_T^2}{2M^2} H_1^\perp(z, z^2\mathbf{k}_T^2)$$

Collins function is link independent though

There is no Burkardt sum rule for $h_1^\perp$, nor a Schäfer-Teryaev sum rule for $D_{1T}^\perp$

Conclusions

- the transverse spin decomposition is not interesting when it comes to the *values* of the contributions (the same as in the longitudinal sum rule), but it shows that the OAM and hence the full decomposition is not simply rotationally invariant, even after x integration

- validity of Burkhardt-Cottingham and any other sum rule crucially depends on the small-x behavior of functions, which is not a fully settled matter yet

- extrapolation of the Bessel-weighted Sivers shift on the lattice tells us about the Qiu-Sterman function

- the transverse moment of the Sivers function may require regularization that may not preserve the Burkardt sum rule automatically

- Node in Sivers and Qiu-Sterman function *not* ruled out yet, but probably not needed to satisfy Burkardt sum rule

Main message: transverse spin sum rules are nontrivial and force us to think about the limitations of the theoretical description, which is very useful beyond the sum rules