The R-matrix Method in Nuclear Physics

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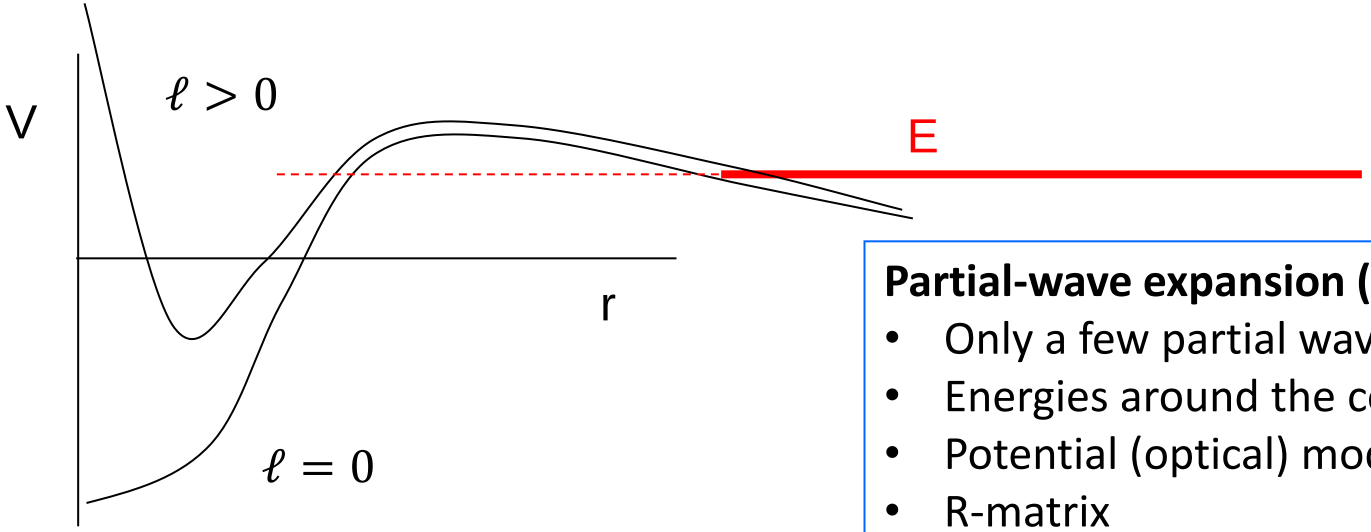
1. Introduction

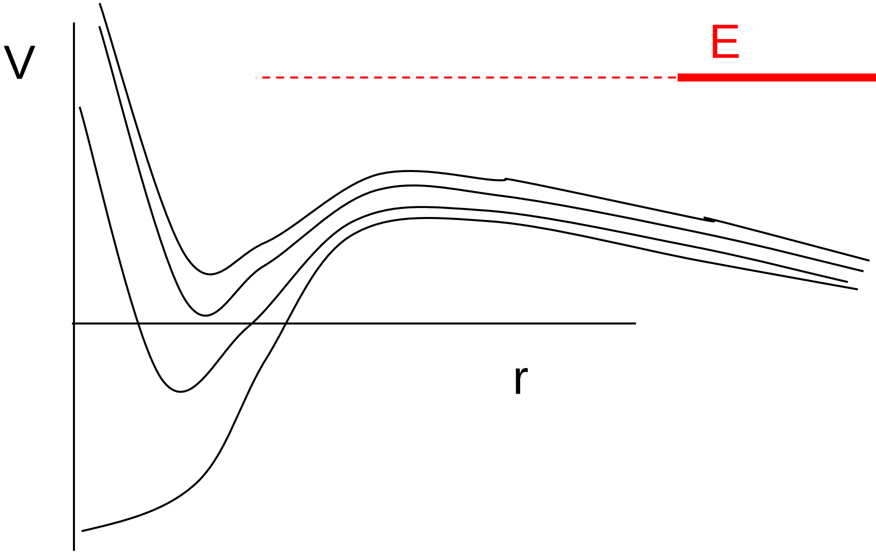
General context:

- Two-body systems
- Low energies (E ≲ Coulomb barrier), few open channels (one)
- Low masses (A ≲ 15-20)
- Low level densities (≲ a few levels/MeV)
- Reactions with neutrons AND charged particles



Partial-wave expansion (low E)
- Only a few partial waves contribute
- Energies around the coulomb barrier
- Potential (optical) model
- R-matrix
- DWBA
- CDCC, etc..

1. Introduction



No partial-wave expansion (high E)

- Many partial waves
- Glauber method
- Eikonal method, etc.

1. Introduction

Different types of reactions

1. Elastic collision : entrance channel=exit channel

$A+B \rightarrow A+B$: Q=0

covered here

2. Inelastic collision (Q≠0)

$A+B \rightarrow A^*+B$ (A^*=excited state)
$A+B \rightarrow A+B^*$
etc..

3. Transfer reactions

$A+B \rightarrow C+D$
$A+B \rightarrow C+D+E$
etc…

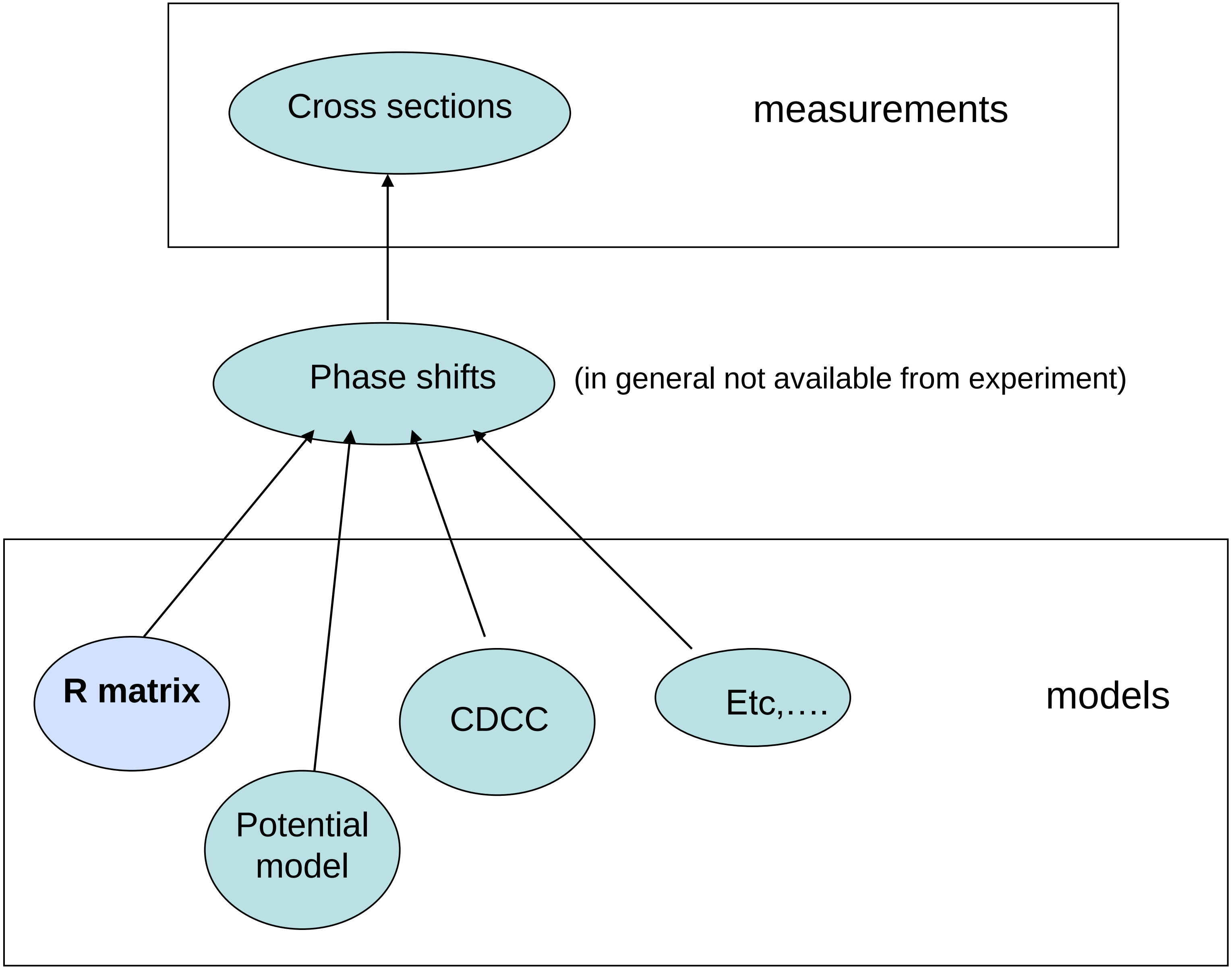
4. Radiative capture reactions
$A+B \rightarrow C + \gamma$

5. Breakup

NOT covered here

1. Introduction

Principles of the partial-wave expansion



Cross sections

measurements

Phase shifts

(in general not available from experiment)

R matrix

Potential model

CDCC

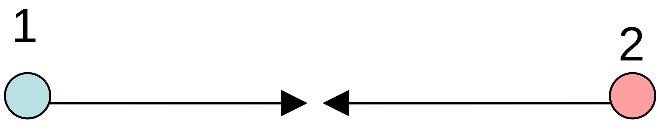
Etc,….

models

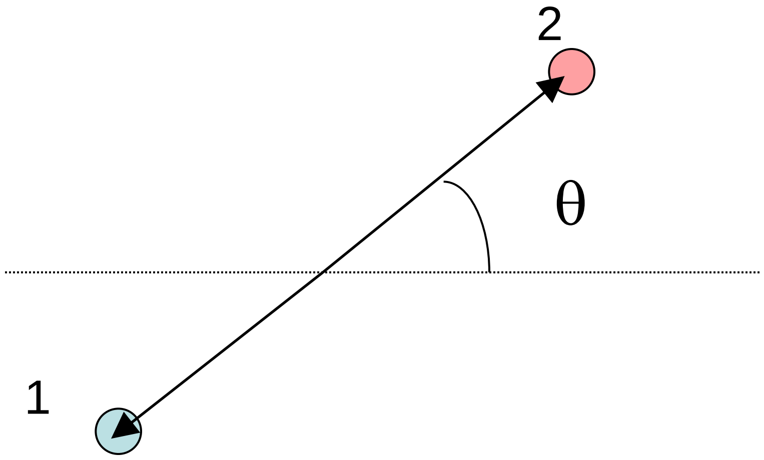
2. Collision theory: elastic scattering

Assumptions:

- elastic scattering
- no internal structure
- no Coulomb
- spins zero



Before collision



After collision

Center-of-mass system

2. Collision theory: elastic scattering

A. Scattering wave functions

Schrödinger equation: V(r)=interaction potential

$$H\Psi(\boldsymbol{r}) = \left(-\frac{\hbar^2}{2\mu}\Delta + V(\boldsymbol{r})\right)\Psi(\boldsymbol{r}) = E\Psi(\boldsymbol{r})$$

- E>0: scattering states
- Assumption: the potential is real and decreases faster than 1/r
- center-of-mass removed
- At large distances : V(r)→ 0

  2 independent solutions : $\Psi(\boldsymbol{r}) \rightarrow A\left(e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + f(\theta)\frac{e^{ikr}}{r}\right)$

Incoming plane wave

Outgoing spherical wave

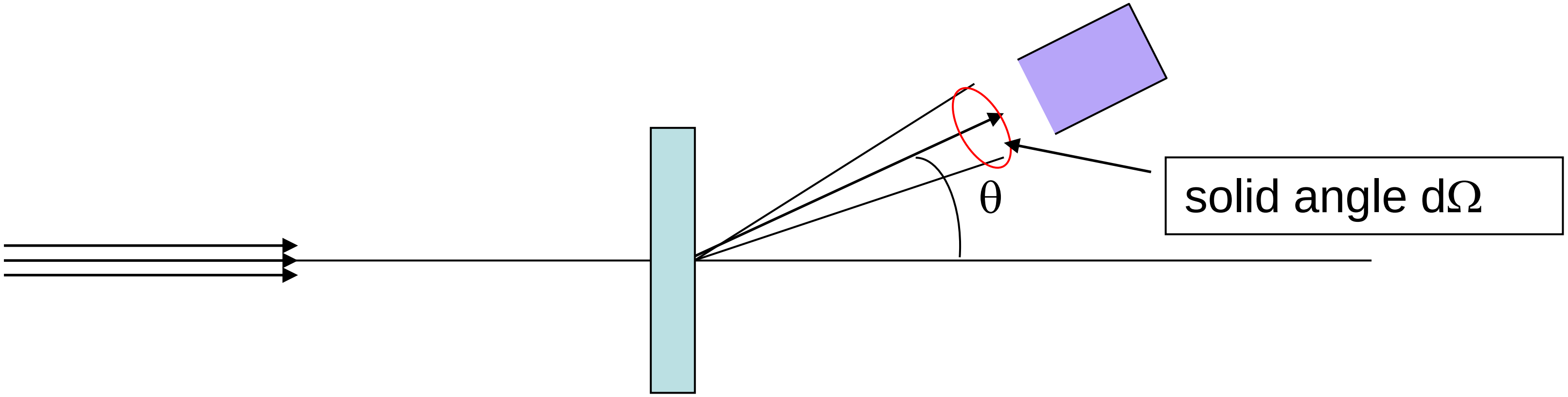
where: k=wave number: $k^2=2mE/\hbar^2$

A =amplitude (scattering wave function is not normalized)

$f(\theta)$ =scattering amplitude (length)

2. Collision theory: elastic scattering

B. Cross sections



Cross section

$$\boxed{\frac{d\sigma}{d\Omega} = |f(\theta)|^2, \quad \sigma = \int \frac{d\sigma}{d\Omega} d\Omega}$$

- Cross section obtained from the asymptotic part of the wave function
- "Direct" problem: determine σ from the potential
- "Inverse" problem : determine the potential V from σ
- Angular distribution: E fixed, θ variable
- Excitation function: θ variable, E fixed,

2. Collision theory: elastic scattering

C. Solving the Schrödinger equation for E>0

$$H\Psi(\boldsymbol{r}) = \left(-\frac{\hbar^2}{2\mu}\Delta + V(\boldsymbol{r})\right)\Psi(\boldsymbol{r}) = E\Psi(\boldsymbol{r})$$

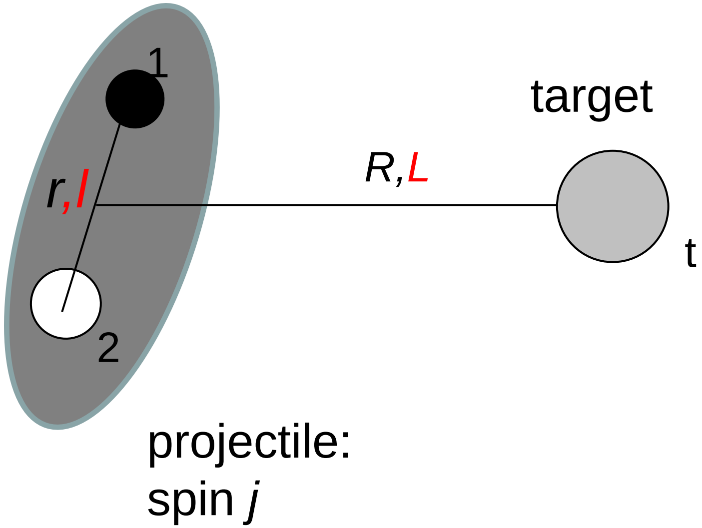
with $\Psi(\boldsymbol{r}) \rightarrow A\left(e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + f(\theta)\frac{e^{ikr}}{r}\right)$ (*z* along the beam axis)

Several methods:

- Here: Two-body Potential model (→ elastic scattering only)
  R-matrix
- *DWBA*: Distorted-Wave Born Approximation
  Different potentials in the entrance and exit channels → transfer

2. Collision theory: elastic scattering

- *CDCC: Coupled-Channel Discretized Continuum*



$$H = H_0 + T_R + V_{t1}(\boldsymbol{R} + \frac{A_2}{A_p}\boldsymbol{r}) + V_{t2}(\boldsymbol{R} - \frac{A_1}{A_p}\boldsymbol{r})$$

- projectile described by a 2 (or 3) cluster structure
- System of coupled-channel equations
- Elastic, inelastic, breakup, etc.

- *Formal theory: Lippman-Schwinger equation*

$f(\theta) = -\frac{2\mu}{4\pi\hbar^2}\int \exp(-ikr'\cos\theta)V(\mathbf{r}')\psi(\mathbf{r}')d\mathbf{r}'$ equivalent to the Schrodinger equation

➔ approximations

- ○ Born approximation : $\Psi(\boldsymbol{r}) = \exp(i\boldsymbol{k}\cdot\boldsymbol{r})$
- ○ Eikonal approximation $\Psi(\boldsymbol{r}) = \exp(i\boldsymbol{k}\cdot\boldsymbol{r})\widehat{\Psi}(\boldsymbol{r})$

3. Phase-shift method: potential model

a. Definitions, cross sections

Simple conditions: neutral systems
spins 0
single-channel

b. Extension to charged systems

c. Extension to multichannel problems

d. Low energy properties

e. General calculation

f. Optical model

3. Phase-shift method: Definition, cross section

The wave function is expanded as

$$\Psi(\boldsymbol{r}) = \sum_{\ell,m} \frac{u_\ell(r)}{r} Y_\ell^m(\Omega_r)\, Y_\ell^{m*}(\Omega_k)$$

which provides the Schrödinger equation for each partial wave ($\Omega_k = 0 \ \rightarrow m = 0$)

$$-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2}\right) u_\ell + V(r) u_\ell = E u_\ell$$

To be solved for any potential (real)

For r→∞

$$\begin{cases} V(r) = 0 \\ u_\ell'' - \frac{\ell(\ell+1)}{r^2} u_\ell + k^2 u_\ell = 0 \ \text{ Bessel equation } \ \rightarrow u_\ell(r) = r j_\ell(kr), r n_\ell(kr) \end{cases}$$

Remarks: at low energies: few partial waves in the expansion

al small r: $u_\ell(r) \rightarrow r^{\ell+1}$

3. Phase-shift method: Definition, cross section

For small x
$$j_l(x) \to \frac{x^l}{(2l+1)!!}$$
$$n_l(x) \to -\frac{(2l-1)!!}{x^{l+1}}$$

For large x
$$j_l(x) \to \frac{1}{x}\sin(x - l\pi/2)$$
$$n_l(x) \to -\frac{1}{x}\cos(x - l\pi/2)$$

Examples: $$j_0(x) = \frac{\sin(x)}{x}, \qquad n_0(x) = -\frac{\cos(x)}{x}$$

At large distances: u_ℓ is a linear combination of $rj_\ell(kr)$ and $rn_\ell(kr)$

$$u_\ell(\mathrm{r}) \to Cr(j_\ell(kr) - \tan\delta_\ell \times n_\ell(kr))$$

With δ_ℓ = phase shift (information about the potential): If V=0 → $\delta_\ell = 0$

3. Phase-shift method: Definition, cross section

Derivation of the elastic cross section

- Identify the asymptotic behaviours

$$\Psi(\boldsymbol{r}) \rightarrow A\left(e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + f(\theta)\frac{e^{ikr}}{r}\right)$$

$$\Psi(\boldsymbol{r}) \rightarrow \sum_{\ell} C_{\ell}(j_{\ell}(kr) \ - \tan \delta_{\ell} \times n_{\ell}(kr))Y_{\ell}^{0}(\Omega_{r})\sqrt{\frac{2\ell+1}{4\pi}}$$

- Provides coefficients C_{ℓ} and scattering amplitude $f(\theta)$

$$f(\theta, E) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)(\exp(2i\delta_{\ell}(E)) - 1)\, P_{\ell}(\cos\theta)$$

$$\frac{d\sigma(\theta,E)}{d\Omega} = |f(\theta, E)|^{2}$$

- Integrated cross section (neutral systems only)

$$\sigma = \frac{\pi}{k^{2}}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^{2}\delta_{\ell}$$

- In practice, the summation over ℓ is limited to some ℓ_{max}

3. Phase-shift method: Definition, cross section

$$\frac{d\sigma(\theta,E)}{d\Omega} = |f(\theta,E)|^2 \text{ with } f(\theta,E) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)(\exp(2i\delta_\ell(E)) - 1)\, P_\ell(\cos\theta)$$

➔ factorization of the dependences in E and θ

low energies: small number of ℓ values ($\delta_\ell \to 0$ when ℓ increases)

high energies: large number (→ alternative methods)

General properties of the phase shifts

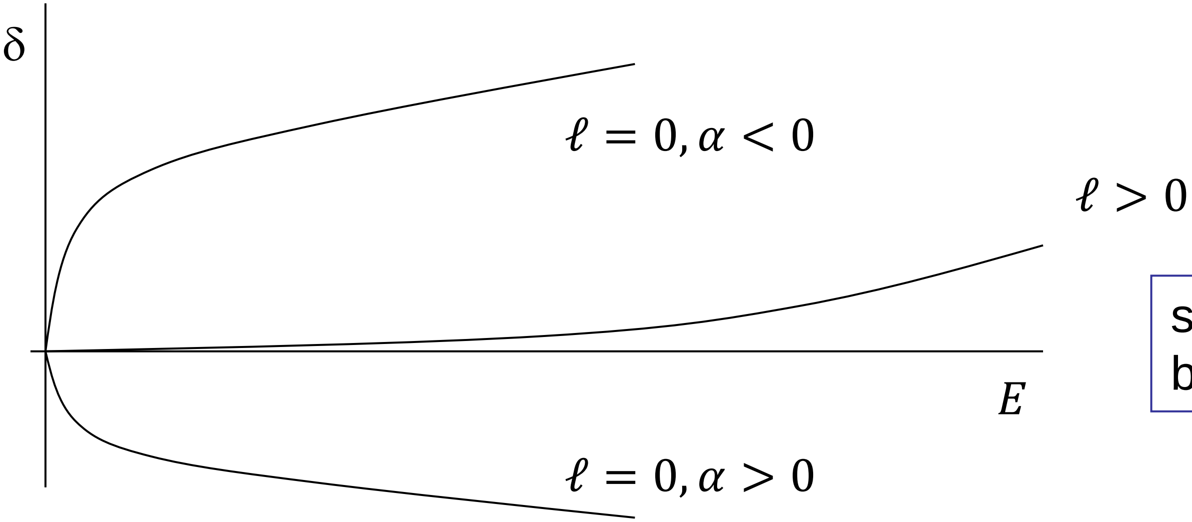
1. The phase shift (and all derivatives) are continuous functions of E
2. The phase shift is known within nπ: $\exp 2i\delta = \exp(2i(\delta + n\pi))$
3. Levinson theorem
    - $\delta_\ell(E = 0)$ is arbitrary
    - $\delta_\ell(0) - \delta_\ell(\infty) = \mathrm{N}\pi$, where N is the number of bound states in partial wave ℓ
    - Example: p+n, $\ell = 0$: $\delta_0(0) - \delta_0(\infty) = \pi$ (bound deuteron)
      $\ell = 1$: $\delta_1(0) - \delta_1(\infty) = 0$ (no bound state for $\ell = 1$)

3. Phase-shift method: Low-energy properties

One defines $\gamma = \frac{1}{u(a)}\left(\frac{du}{dr}\right)_{r=a}$ =logarithmic derivative at *r*=*a* (*a* large)

Then, for very low E (neutral system): $\tan\delta_\ell \approx \frac{(ka)^{2\ell+1}}{(2\ell+1)!!(2\ell-1)!!}\left[\frac{\ell-\gamma a}{\gamma a+\ell+1}\right]$

For $\ell = 0$: $\alpha = -\lim_{k\to 0}\frac{\tan\delta_0(k)}{k} =$ scattering length



strong difference between $\ell = 0$ and $\ell > 0$

Generalization $k\cot\delta_0(k) = -\frac{1}{\alpha} + \frac{1}{2}r_e k^2 + \ldots$

α=scattering length

r_e=effective range

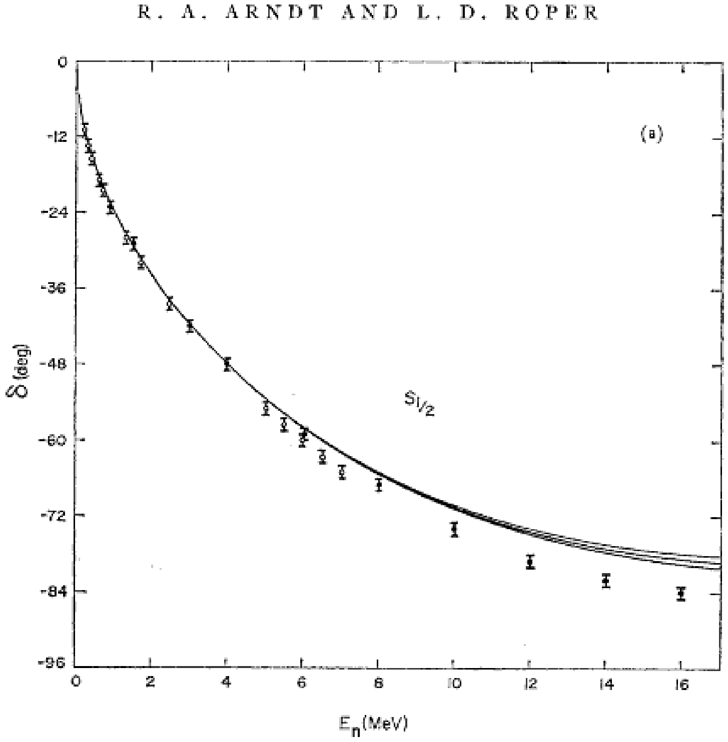
3. Phase-shift method: Low-energy properties

Cross section at low E: $$\sigma = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell \xrightarrow[k\to 0]{} 4\pi\alpha^2 \text{ (isotropic)}$$

In general (**neutrons**): $$\delta \sim k^{2\ell+1}$$

$$\sigma \sim k^{4\ell}$$

Thermal neutrons (T=300K, E=25 meV): only ℓ=0 contributes



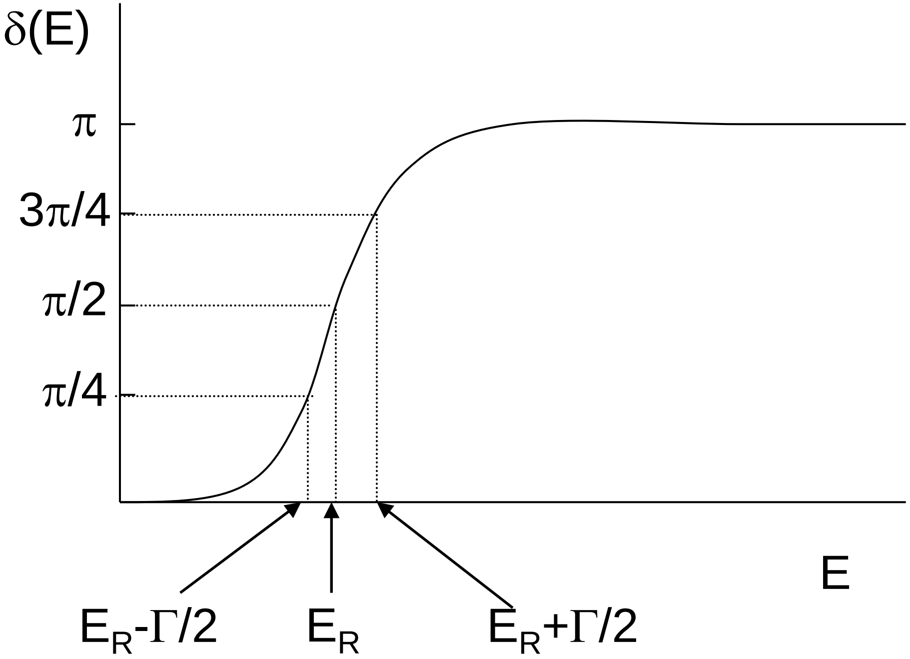
example : α+n phase shift $\ell = 0$

At low E: δ~-kα with $\alpha>0$ ➔ replusive

3. Phase-shift method: resonances

Resonances: $\delta_R(E) \approx \arctan \frac{\Gamma}{2(E_R - E)}$ =Breit-Wigner approximation

E_R=resonance energy
Γ=resonance width



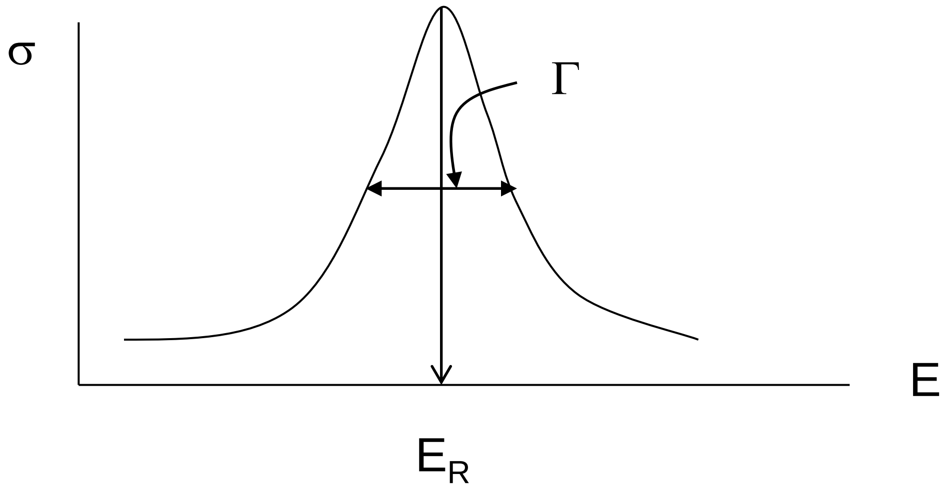
- Narrow resonance: Γ small
- Broad resonance: Γ large

3. Phase-shift method: resonances

Cross section

$\sigma(E) = \frac{\pi}{k^2} \sum_\ell (2\ell + 1) |\exp(2i\delta_\ell) - 1|^2$ maximum for $\delta = \frac{\pi}{2}$

Near the resonance: $\sigma(E) \approx \frac{4\pi}{k^2} (2\ell_R + 1) \frac{\Gamma^2/4}{(E_R - E)^2 + \Gamma^2/4}$, where ℓ_R=resonant partial wave



In practice:

- Peak not symmetric (Γ depends on E)
- « Background » neglected (other ℓ values)
- Differences with respect to Breit-Wigner

3. Phase-shift method: resonances

Narrow vs broad resonances

Comparison of 2 caractrristic times

a. Resonance lifetime $\tau_R = \hbar/\Gamma$

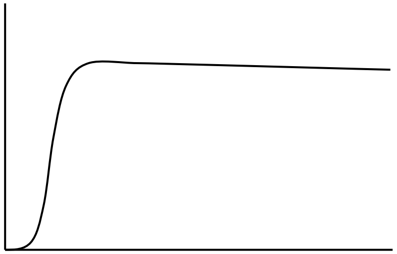
b. Interaction time (no resonance): $\tau_{NR} = d/v$

Example1: ^{12}C+p

Resonance properties: E_R=0.42 MeV, Γ=32 keV → lifetime: τ_R = ~$2\text{x}10^{-20}$ s

Interaction range d~10 fm →interaction time τ_{NR}~$1.1\text{x}10^{-21}$ s

→ narrow resonance

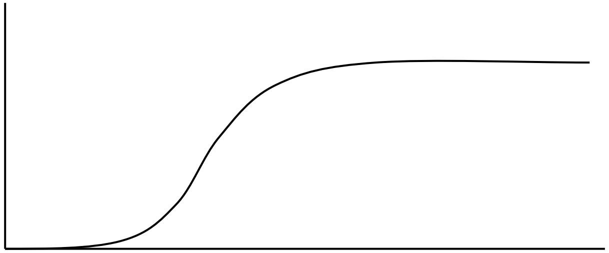


Example 2: α+p

Resonance properties: E_R=1.72 MeV, Γ=1.2 MeV → lifetime: τ_R = ~$6\text{x}10^{-22}$ s

Interaction range d~10 fm →interaction time τ_{NR}~$5\text{x}10^{-22}$ s

→ broad resonance



3. Phase-shift method: Generalization to the Coulomb potential

The asymptotic behaviour $\Psi(\boldsymbol{r}) \to \exp(i\boldsymbol{k}\cdot\boldsymbol{r}) + f(\theta)\frac{\exp(ikr)}{r}$

Becomes: $\Psi(\boldsymbol{r}) \to \exp(i\boldsymbol{k}\cdot\boldsymbol{r} + i\eta\ln(\boldsymbol{k}\cdot\boldsymbol{r} - kr)) + f(\theta)\frac{\exp(i(kr-\eta\ln 2kr))}{r}$

With $\eta = \frac{Z_1 Z_2 e^2}{\hbar v}$=Sommerfeld parameter, v=velocity

Bessel equation: $\left(\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} + k^2\right)u_\ell = 0$

$$u_\ell(r) \to rA(j_\ell(kr) - \tan\delta_\ell\, n_\ell(kr))$$

Coulomb equation: $\left(\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - 2\frac{\eta k}{r} + k^2\right)u_\ell = 0$

Solutions: F_l(h,kr): regular, G_l(h,kr): irregular

Ingoing, outgoing functions: $I_l = G_l - iF_l$, $O_l = G_l + iF_l$

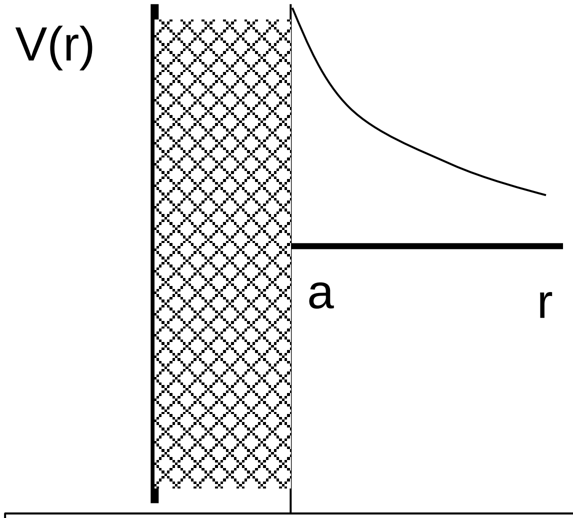
$$u_\ell(r) \to A(F_\ell(\eta,kr) + \tan\delta_\ell\, G_\ell(\eta,kr))$$

$$\to B(\cos\delta_\ell\, F_\ell(\eta,kr) + \sin\delta_\ell\, G_\ell(\eta,kr))$$

$$\to C(I_\ell(\eta,kr) - U_\ell O_\ell(\eta,kr))$$

3. Phase-shift method: Generalization to the Coulomb potential

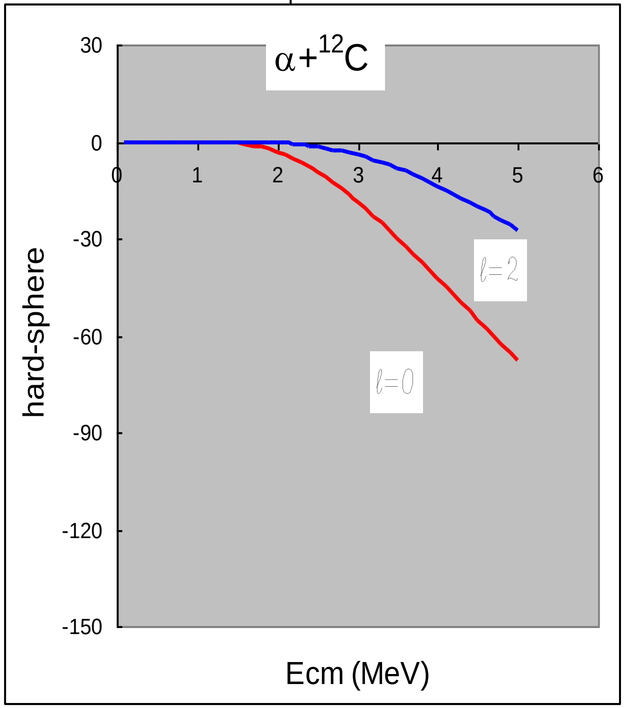
Example: hard sphere

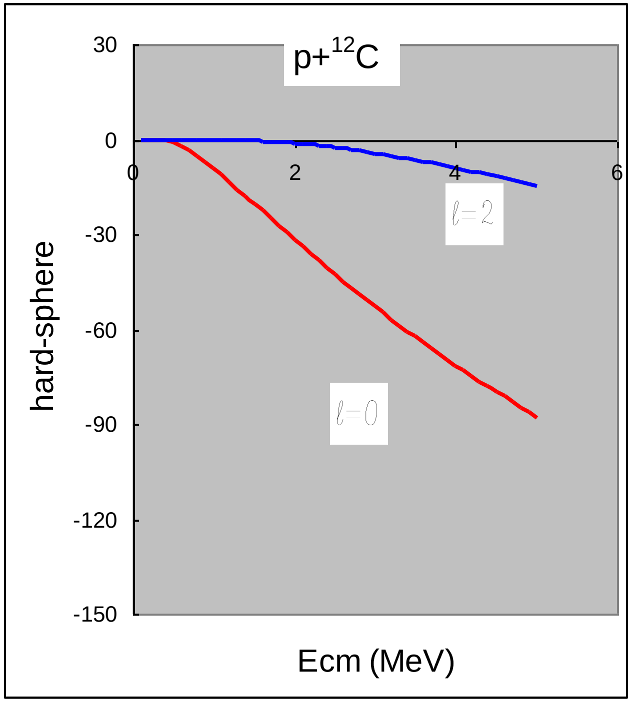


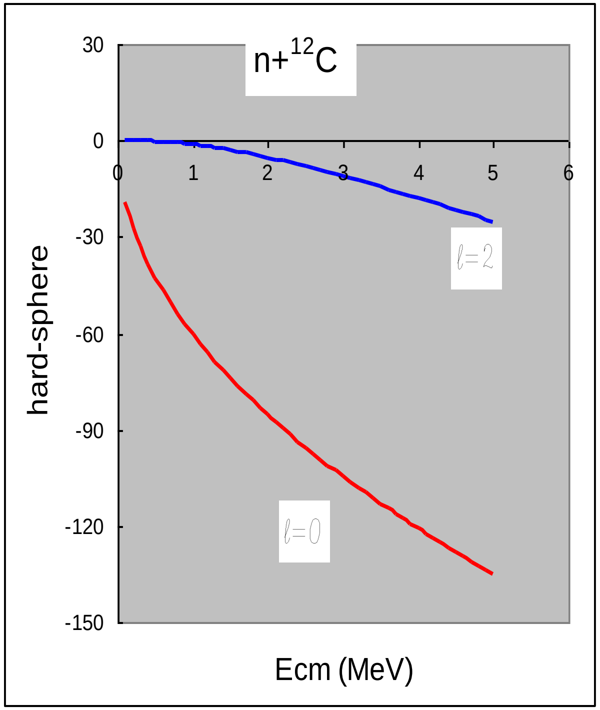
$$V(r) = \frac{Z_1 Z_2 e^2}{r} \text{ for } r > a$$
$$V(r) = \infty \text{ for } r < a$$

phase shift: $u_\ell(a) = F_\ell(\eta, ka) + \tan\delta_\ell\, G_\ell(\eta, kr) = 0$

$$\rightarrow \tan\delta_\ell = -\frac{F_\ell(\eta,ka)}{G_\ell(\eta,ka)}$$







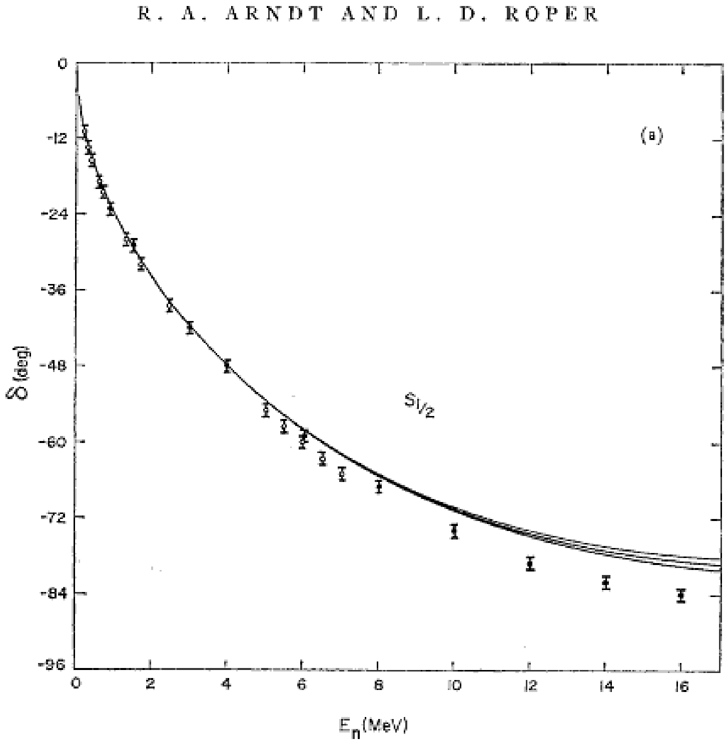
3. Phase-shift method: Generalization to the Coulomb potential

Special case: Neutrons, with $\ell = 0$:

$F_0(x)=\sin(x)$, $G_0(x)=\cos(x)$

→ $\delta=-ka$

example : α+n phase shift $\ell = 0$



➔ hard sphere is a good approximation

3. Phase-shift method: Generalization to the Coulomb potential

Elastic cross section with Coulomb:

$$\frac{d\sigma}{d\Omega} = | f(\theta) |^2, \text{ with } f(\theta) = \frac{1}{2ik} \sum_l (2l+1)(\exp(2i\delta_l) - 1) P_l(\cos\theta)$$

Still valid, but converges very slowly.

$$\delta_l = \delta_l^N + \delta_l^C$$

$$f(\theta) = \frac{1}{2ik} \sum_l (2l+1)[\exp(2i\delta_l) - 1] P_l(\cos\theta)$$

$$= \frac{1}{2ik} \sum_l (2l+1)[\underbrace{\exp(2i\delta_l) - \exp(2i\delta_l^C)}_{\boxed{\text{Nuclear}}} + \underbrace{\exp(2i\delta_l^C) - 1}_{\boxed{\text{Coulomb: exact}}}] P_l(\cos\theta)$$

$$= f_N(\theta) + f_C(\theta)$$

$$f_C(\theta) = \frac{-\eta}{2k \sin^2 \theta/2} \exp(-i\eta \log(\sin^2 \theta/2)) = \text{Coulomb amplitude}$$

$$\frac{d\sigma_C}{d\Omega} = | f_C(\theta) |^2$$

= Coulomb/Rutherford cross section:

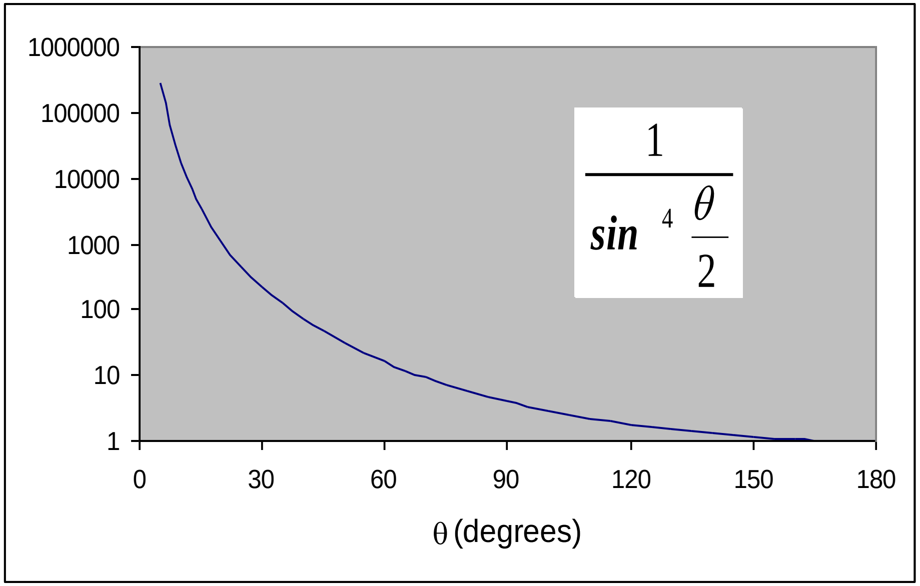
diverges for $\theta=0$

increases at low energies

3. Phase-shift method: Generalization to the Coulomb potential

$$f(\theta) = f_C(\theta) + f_N(\theta)$$

- $f_C(\theta)$: Coulomb part: exact
- $f_N(\theta)$: nuclear part: converges rapidly



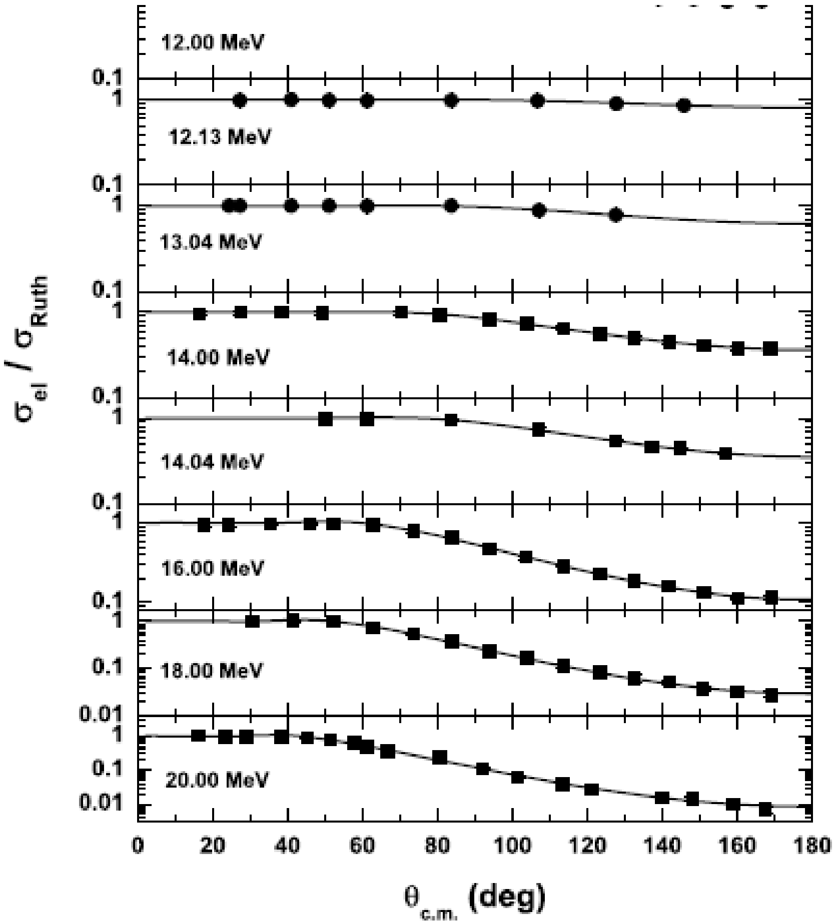
$$\frac{d\sigma_C}{d\Omega} = |f_C(\theta)|^2 \sim \frac{1}{E^2}\frac{1}{\sin^4\frac{\theta}{2}}$$

- $\frac{d\sigma_C}{d\Omega}$ =Rutherford cross section
- The total Coulomb cross section is not defined (diverges)
- Coulomb is dominant at small angles
  → used to normalize data
- Increases at low energies
- Minimum at $\theta=180^o$ → nuclear effect maximum

3. Phase-shift method: Generalization to the Coulomb potential

Cross section: $\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = |f^C(\theta) + f^N(\theta)|^2$

- Coulomb amplitude strongly depends on the angle → $\frac{d\sigma/_{d\Omega}}{d\sigma_C/_{d\Omega}}$



^{6}Li+^{58}Ni system

- $E_{cm} = \frac{58}{64} E_{lab}$
- Coulombienne barrier

$$E_B \sim \frac{3 * 28 * 1.44}{7} \sim 17 \text{ MeV}$$

- Below the barrier: $\sigma \sim \sigma_C$
- Above E_B : σ is different from σ_C

3. Phase-shift method: Extension to multichannel problems

- One channel: phase shift $\delta \rightarrow U=\exp(2i\delta)$
- Multichannel: *collision matrix* U_{ij}, (symmetric , unitary) with i,j=channels
- Good quantum numbers J=total spin π=total parity
- Channel *i* characterized by $I=I_1 \oplus I_2$ =channel spin
  $J=I \oplus \ell$ ℓ =angular momentum
- Selection rules: $|I_1 - I_2| \leq I \leq I_1 + I_2$
  $|\ell - I| \leq J \leq \ell + I$
  $\pi=\pi_1 * \pi_2 * (-1)^{\ell}$

Example of quantum numbers

$\alpha + {}^3$He $\alpha=0^+$, ^{3}He=1/2$^+$

J	*I*	ℓ	size
1/2+	1/2	0, ~~1~~	1
1/2-	1/2	~~0~~, 1	1
3/2+	1/2	~~1~~, 2	1
3/2-	1/2	1, ~~2~~	1

p+^{7}Be ^{7}Be=3/2$^-$, p=1/2$^+$

J	*I*	ℓ	size
0+	1 2	1 ~~2~~	1
0-	1 2	~~1~~ 2	1
1+	1 2	~~0~~, 1, ~~2~~ 1, ~~2~~, 3	3
1-	1 2	0, ~~1~~, 2 ~~1~~, 2, ~~3~~	3

3. Phase-shift method: Extension to multichannel problems

Cross sections

One channel: $\frac{d\sigma}{d\Omega} = |f(\theta)|^2$, with $f(\theta) = \frac{1}{2ik}\sum_l (2l+1)(\exp(2i\delta_l)-1)P_l(\cos\theta)$

Multichannel $\frac{d\sigma}{d\Omega} = \sum_{K_1,K_2,K_1',K_2'} |f_{K_1K_2,K_1'K_2'}(\theta)|^2$

With: K_1,K_2=spin orientations in the entrance channel

K'_1,K'_2=spin orientations in the exit channel

$$f_{K_1K_2,K_1'K_2'}(\theta) = \sum_{J,\pi}\sum_{lI,l'I'} \dots U^{J\pi}_{lI,l'I'}\; Y_{l'}(\theta,0)$$

Collision matrix

- generalization of δ: $U_{ij}=\eta_{ij}\exp(2i\delta_{ij})$
- determines the cross section

3. Phase-shift method: General calculation

For some potentials: analytic solution of the Schrödinger equation

In general: no analytic solution → numerical approach

$$-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}u_\ell(r) + (V(r) - E)u_\ell(r) = 0$$

with: $$V(r) = V_N(r) + \frac{Z_1 Z_2 e^2}{r} + \frac{\hbar^2}{2\mu}\frac{\ell(\ell+1)}{r^2}$$

$$u_\ell(r) \rightarrow F_\ell(kr,\eta)\cos\delta_\ell + G_\ell(kr,\eta)\sin\delta_\ell$$

Numerical solution : discretization N points, with mesh size h

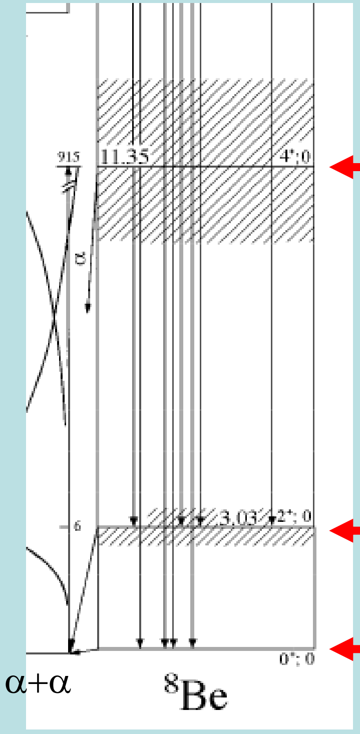
- $u_\ell(0)=0$
- $u_\ell(h)=1$ (or any constant)
- $u_\ell(2h)$ is determined numerically from $u_\ell(0)$ and $u_\ell(h)$ (Numerov algorithm)
- $u_\ell(3h)$,… $u_\ell(Nh)$
- for large r: matching to the asymptotic behaviour → phase shift

Bound states: same idea

3. Phase-shift method: General calculation

Example: α+α

Experimental spectrum of ^{8}Be

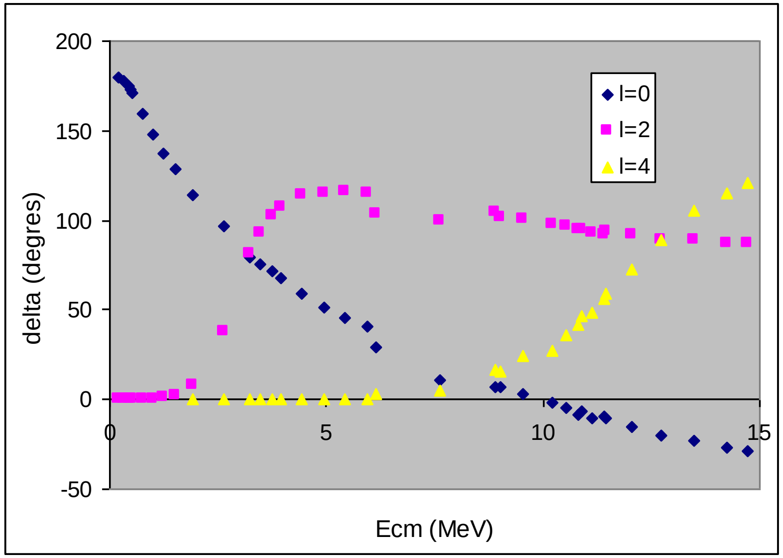


4^+
E~11 MeV
Γ~3.5 MeV

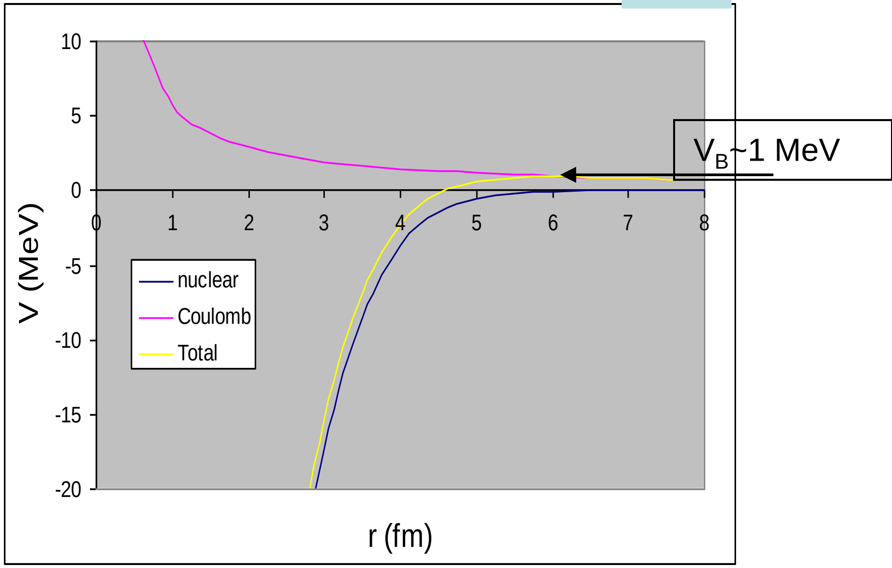
2^+
E~3 MeV
Γ~1.5 MeV

0^+
E=0.09 MeV
Γ=6 eV

Experimental phase shifts

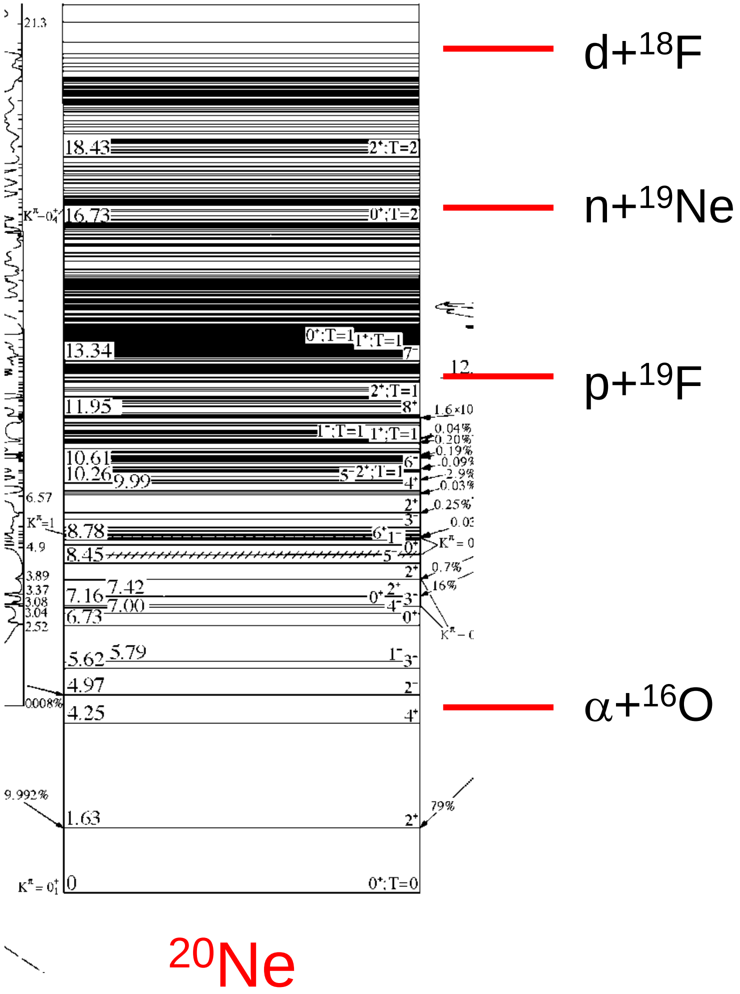


Potential: $V_N(r)=-122.3*\exp(-(r/2.13)^2)$



3. Phase-shift method: Optical model

Goal: to simulate absorption channels



High energies:

- many open channels
- strong absorption
- potential model extended to complex potentials (« optical »)

Phase shift is complex: $\delta = \delta_R + i\delta_I$

collision matrix: $U = \exp(2i\delta) = \eta \exp(2i\delta_R)$

where $\eta = \exp(-2\delta_I) < 1$

Elastic cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{4k^2}\left|\sum_\ell (2\ell+1)(\eta_\ell \exp(2i\delta_\ell) - 1)P_\ell(\cos\theta)\right|^2$$

Reaction cross section:

$$\sigma = \frac{\pi}{k^2}\sum_\ell (2\ell+1)(1-\eta_\ell^2)$$

4. The R-matrix Method: general formalism

A. Aims of the R-matrix method

1. To solve the Schrödinger equation (E>0 or E<0)=“calculable R-matrix”
    1. **Potential model**
    2. 3-body scattering
    3. Microscopic models
    4. Many applications in nuclear and atomic physics

2. To fit cross sections=“phenomenological” R-matrix
    1. Elastic, inelastic → spectroscopic information on resonances
    2. Capture, transfer → nuclear astrophysics

- Main reference: A.M. Lane and R.G. Thomas, Rev. Mod. Phys. 30 (1958) 257
- More ‘’modern’’ reference: P. D., D. Baye, Rep. Prog. Phys. 73 (2010) 036301

4. The R-matrix Method: general formalism

B. Standard variational calculations

- Hamiltonian $(T+V)u = Eu$
- Set of *N* basis functions $\phi_i(r)$ with $u(r) = \sum_i c_i\phi_i(r)$

➔Calculation of $H_{ij} =< \phi_i|\, H|\phi_j >$ over the full space

$N_{ij} =< \phi_i|\phi_j >$

(example: gaussians: $\phi_i(r) = \exp(-(r/a_i)^2)$)

$$H_{ij} =< \phi_i|\, H|\phi_j >= \int_0^\infty \phi_i(r)H\phi_j(r)dr$$

$$N_{ij} =< \phi_i|\phi_j >= \int_0^\infty \phi_i(r)\phi_j(r)dr$$

- Eigenvalue problem : $\sum_j \left(H_{ij} - EN_{ij}\right)c_j = 0 \ \rightarrow$ upper bound on the energy
- But: Functions $\phi_i(r)$ tend to zero $\rightarrow$ not directly adapted to scattering states

4. The R-matrix Method : general formalism

C. Extension to the R-matrix:

includes boundary conditions

a. Hamiltonian $(T+V)u = Eu$

with $V(r) \to \frac{Z_1 Z_2 e^2}{r}$

b. Wave functions

Set of *N* basis functions $\phi_i(r)$:

$u_\ell(r) = \sum_{i=1} c_i \phi_i(r)$ valid in a **limited** range

+correct asymptotic behaviour (E>0):

$$u_\ell(r) \to A(I_\ell(kr) - U_\ell O_\ell(kr))$$

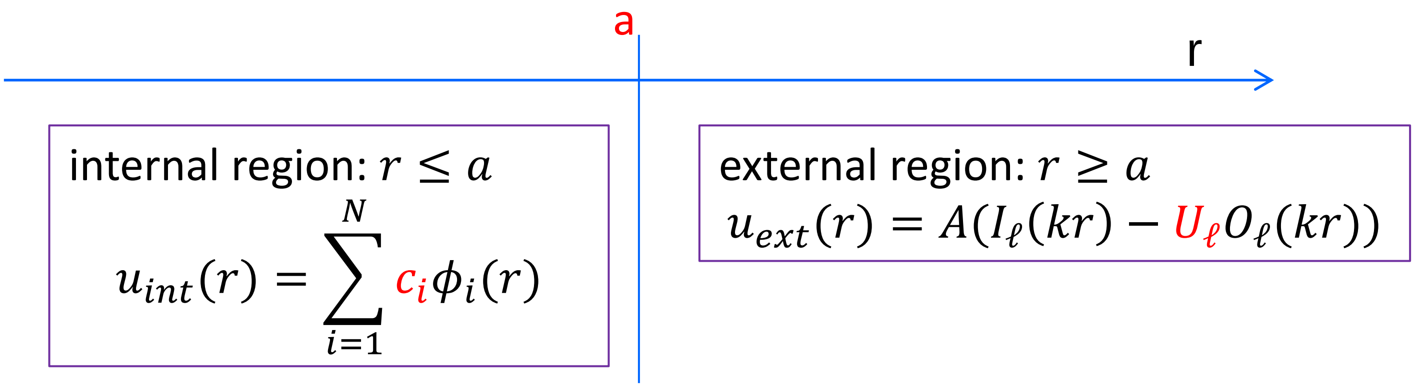
4. The R-matrix Method: general formalism

D. Principles of the R-matrix

- Presented for 2 structureless particles interacting by a potential $V(r)$
  one channel

- N basis functions $\phi_i(r)$: variational calculations: $u(r) = \sum_{i=1}^{N} c_i \phi_i(r)$
  tend to zero at large distances
  valid at short distances only

- Definition of 2 regions: channel radius a



internal region: $r \leq a$

$$u_{int}(r) = \sum_{i=1}^{N} c_i \phi_i(r)$$

external region: $r \geq a$

$$u_{ext}(r) = A(I_\ell(kr) - U_\ell O_\ell(kr))$$

- Schrödinger equation + Matching at $r = a$
  ➔ collision matrix U_ℓ (~phase shift) and coefficients c_i (wave function)
  !! Channel radius a is not a parameter!!

4. The R-matrix Method: general formalism

- matrix elements over the internal region

$$H_{ij} =< \phi_i|\, H|\phi_j >_{int}= \int_0^a \phi_i(r)(T+V)\phi_j(r)dr$$

$$N_{ij} =< \phi_i|\phi_j >_{int}= \int_0^a \phi_i(r)\phi_j(r)dr$$

Problem: the kinetic energy is not hermitian over a finite interval [0,a]

$$\int_0^a \phi_i(r)\frac{d^2}{dr^2}\phi_j(r)dr \neq \int_0^a \phi_j(r)\frac{d^2}{dr^2}\phi_i(r)dr$$

→ Bloch operator

$$\mathcal{L}(L) = \frac{\hbar^2}{2\mu a}\delta(r-a)\left(\frac{d}{dr}-\frac{L}{r}\right)r$$

L=arbitrary constant (L=0 in most cases)

role of the surface operator $\mathcal{L}(L)$: makes $\mathrm{T}+\mathcal{L}(L)$ hermitian

$$< \phi_i|\, T+\mathcal{L}(L)|\phi_j >_{int}=< \phi_j|\, T+\mathcal{L}(L)|\phi_i >_{int}$$

4. The R-matrix Method: general formalism

E. Collision matrix

- The Schrödinger equation

$$(H-E)u_\ell = 0$$

is replaced by the Bloch-Schrödinger

$$(H-E+\mathcal{L}(L))u_{int} = \mathcal{L}(L)u_{int} = \mathcal{L}(L)u_{ext}$$

- 2nd role of the Bloch operator: ensures $u'_{int}(a) = u'_{ext}(a)$ (for the exact solution)

- We use

$$(H-E+\mathcal{L}(L))u_\ell = \mathcal{L}(L)u_\ell \qquad (1)$$

$$u_{int}(r) = \sum_{i=1}^{N} c_i\phi_i(r) \qquad (2)$$

$$u_{ext}(r) = A(I_\ell(kr) - U_\ell O_\ell(kr)) \qquad (3)$$

$$u_{int}(a) = u_{ext}(a) \qquad (4)$$

With (1),(2),(3) →$\sum_{i=1}^{N} c_i \underbrace{< \phi_j\, |H-E+\mathcal{L}(L)|\phi_i >_{int}}_{\text{matrix } D_{ij}} = < \phi_j\,(r)|\mathcal{L}(L)|u_{ext} >$

→ after inversion, provides coefficients c_i

with (4) →$\sum_{i=1}^{N} c_i\phi_i(a) = A(I_\ell(ka) - U_\ell O_\ell(ka))$

4. The R-matrix Method: general formalism

From these 2 equations, one gets the collision matrix (=scattering matrix)

$$U(E) = \frac{I(ka)}{O(ka)} \frac{1-L^* R(E)}{1-LR(E)}$$

with $R(E) = \frac{\hbar^2 a}{2\mu} \sum_{ij} \phi_i(a) \, (D^{-1})_{ij} \phi_j(a)$ (R-matrix:1x1 for single-channel calculations)

$L = ka \frac{O'(ka)}{O(ka)} = S(E) + iP(E)$ (L is complex)

Factorization of U

$$U(E) = \frac{I(ka)}{O(ka)} \frac{1-L^* R(E)}{1-LR(E)} = \exp(2i\delta) = \exp(2i\delta_{HS}) \exp(2i\delta_R)$$

- Hard-sphere phase shift: $\exp(2i\delta_{HS}) = \frac{I(ka)}{O(ka)} \rightarrow \delta_{HS} = -\,\mathrm{atan} \frac{F(ka)}{G(ka)}$
- R-matrix phase shift: $\exp(2i\delta_R) = \frac{1-L^* R(E)}{1-LR(E)} = \frac{1-SR+iPR}{1-SR-iPR} \rightarrow \delta_R = \mathrm{atan} \frac{PR}{1-SR}$
- δ_{HS} and δ_R depend on a but **the sum should not depend on a**

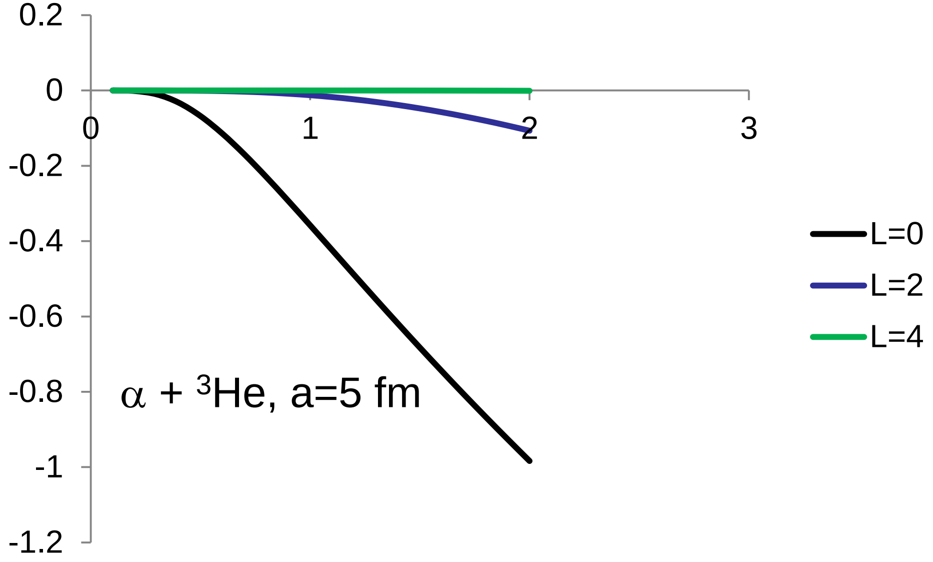
- If the potential is real: R is real and $|U(E)| = 1$

4. The R-matrix Method: general formalism

F. Coulomb functions

- hard-sphere phase shift
- functions S(E) (shift) and P(E) (penetration factor)

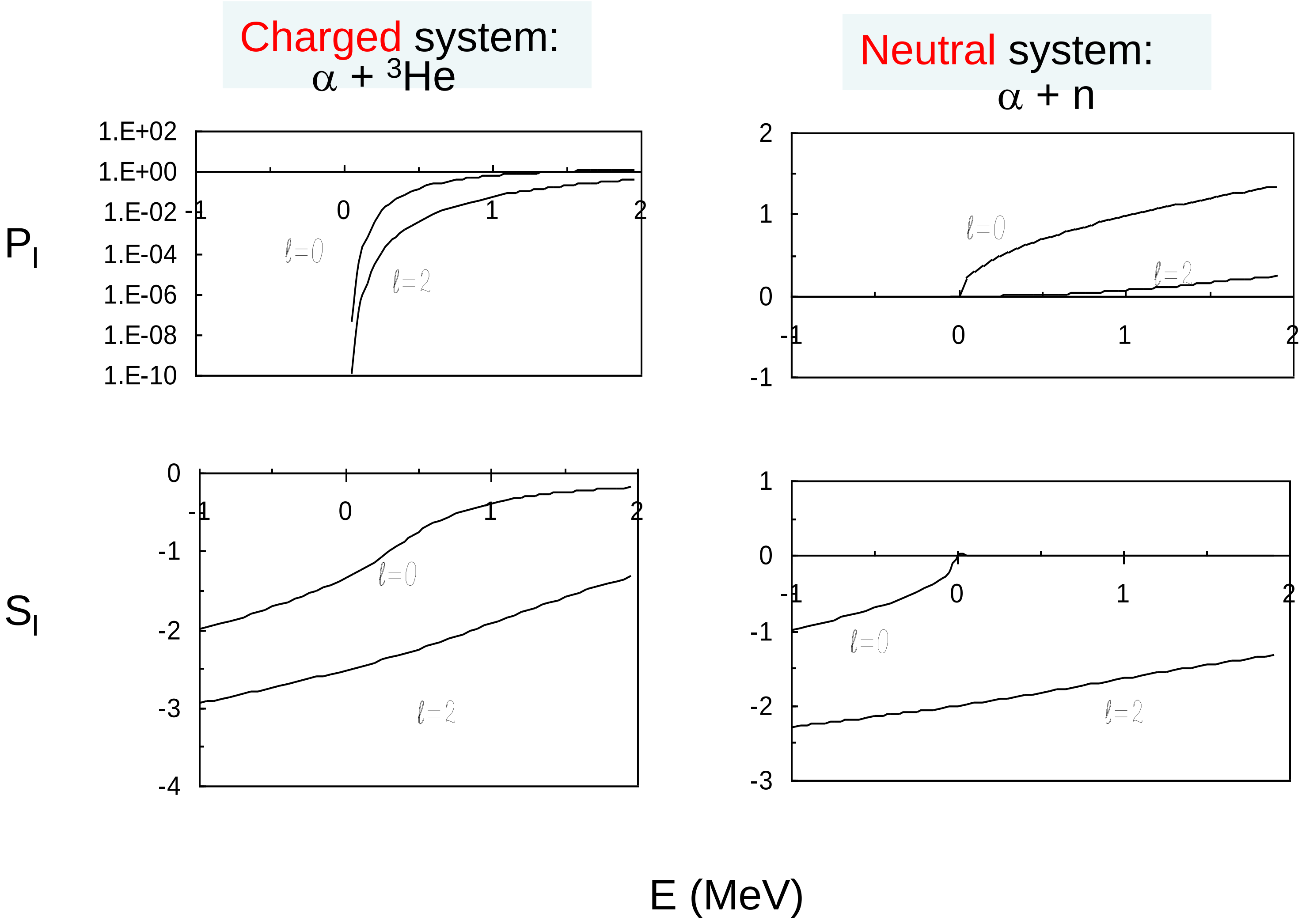
1. hard-sphere phase shift δ_{HS}



4. The R-matrix Method: general formalism

2. shift (S) and penetration (P) factors

From $L = ka\frac{O'(ka)}{O(ka)} = S(E) + iP(E)$



4. The R-matrix Method: general formalism

Procedure (for a given ℓ):

1. Compute matrix elements

$$D_{ij}(E) =< \phi_j|H - E + \mathcal{L}(0)|\phi_i >_{int}$$

In the potential model

$$D_{ij}(E) = \int_0^a \phi_j(r)[\left(-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2}\right) + V(r) + \mathcal{L}(0) - E]\phi_i(r)dr$$

2. Invert matrix D

3. Compute R matrix: $R(E) = \frac{\hbar^2 a}{2\mu}\sum_{ij}\phi_i(a)\,(D^{-1})_{ij}\phi_j(a)$

4. Compute the collision matrix $U(E) = \exp(2i\delta) = \frac{I(ka)}{O(ka)}\frac{1-L^*R(E)}{1-LR(E)}$

Tests: Stability with the channel radius
with the number of basis functions

4. The R-matrix Method: general formalism

G. Alternative formulation of the R-matrix

- The basis functions are taken as the eigenvectors of $(H - E_\lambda + \mathcal{L}(L))\ \phi_\lambda = 0$

  with $\lambda =$ poles: depend on a

- Matrix $D'_{\lambda\lambda} =< \phi_\lambda | H - E + \mathcal{L}(L) | \phi_{\lambda'} > = (E_\lambda - E)\delta_{\lambda\lambda'}$

  → inversion very simple

  → $R(E) = \sum_{\lambda=1}^{N} \frac{\gamma_\lambda^2}{E_\lambda - E}$

  completely equivalent (used in the phenomenological R-matrix)

  with the reduced widths

$$\gamma_\lambda = \left(\frac{\hbar^2}{2\mu a^2}\right)^{1/2} \phi_\lambda(a)$$

  - γ_λ is real and energy independent
  - $\gamma_\lambda \sim$ wave function at the channel radius → measurement of the clustering
  - dimensionless reduced width $\theta^2 = \frac{\gamma^2}{\gamma_W^2}$ where $\gamma_W^2 = \frac{3\hbar^2}{2\mu a^2}$ is the Wigner limit ($\theta^2 < 1$)

5. The calculable R-matrix Method

Aim

- To derive phase shifts (cross sections) from a finite basis
- Exemplified here with the simple **potential model**

Input data

- Potential
- Set of *N* basis functions (here gaussians with different widths)
- Channel radius *a*

Requirements

- *a* large enough : $V_N(a)$~0
- *N* large enough (to reproduce the internal wave functions)
- *N* as small as possible (computer time) → compromise

Tests

- Stability of the phase shift with the channel radius *a*
- Continuity of the derivative of the wave function:

  $u'_{int}(a) = u'_{ext}(a)$ (for the exact solution, not for an approximate solution)

5. The calculable R-matrix Method

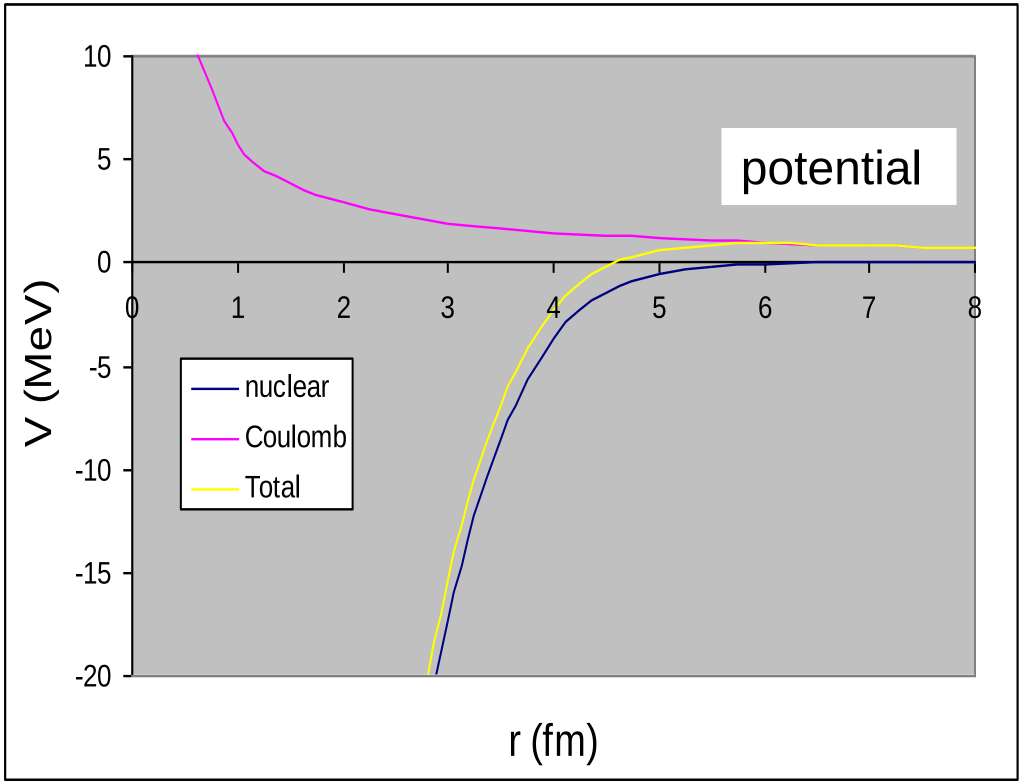
α+α scattering

- potential : $V(r)=-126*\exp(-(r/2.13)^2)$ (Buck potential)

- Basis functions: $\phi_i(r) = r^\ell \exp(-(r/a_i)^2)$

  with $a_i=x_0*a_0^{(i-1)}$ (geometric progression)

  typically $x_0=0.6$ fm, $a_0=1.4$

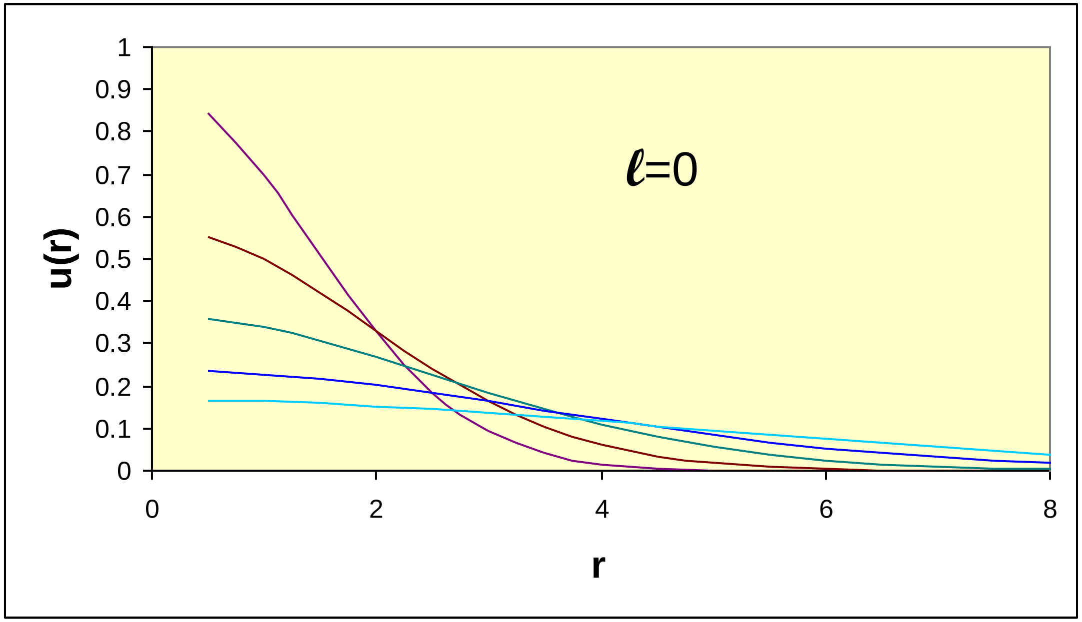


$V_N(a)=0$

→ a > 6 fm

5. The calculable R-matrix Method

Basis functions: $\phi_i(r) = r^\ell \exp(-(r/a_i)^2)$ Gaussians with different widths



$$N_{ij} =< \phi_j \phi_i >_{int}$$
$$= \int_0^a r^{2\ell} \exp\left(-(r/a_i)^2 - \left(r/a_j\right)^2\right) dr$$

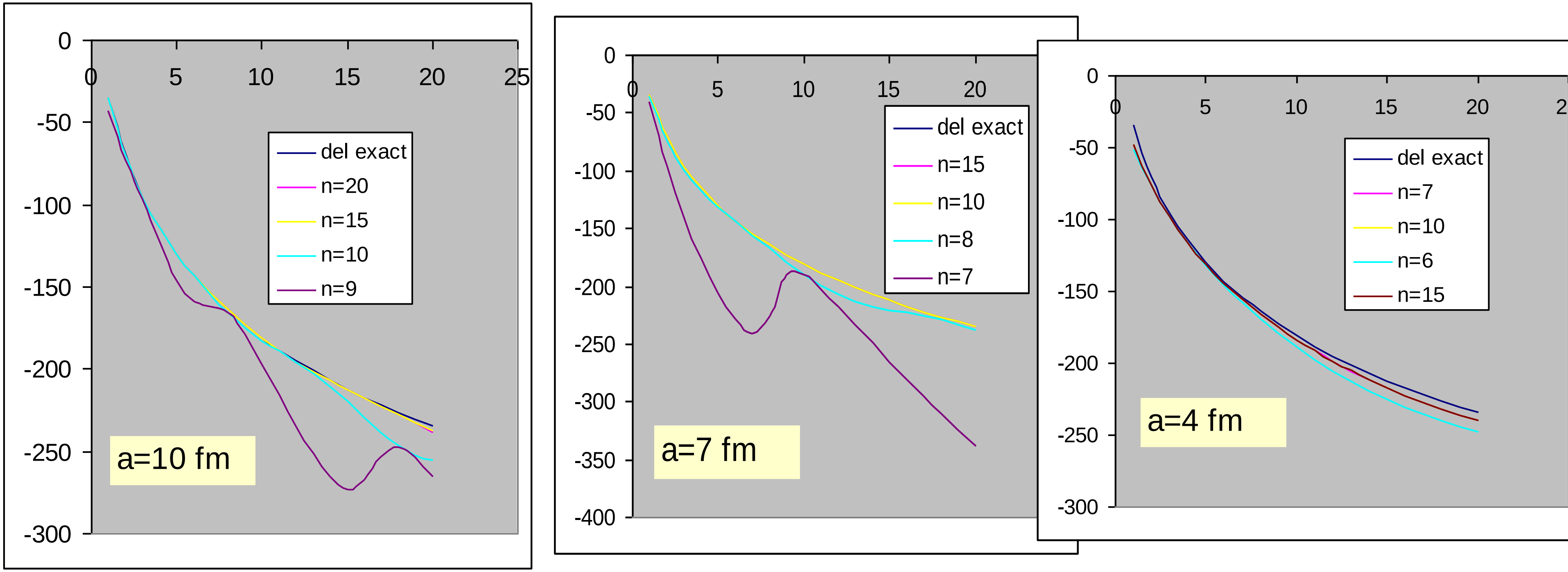
Can be done exactly
(incomplete γ function)

Matrix elements of H can be calculated analytically for gaussian potentials

Other potentials: numerical integration

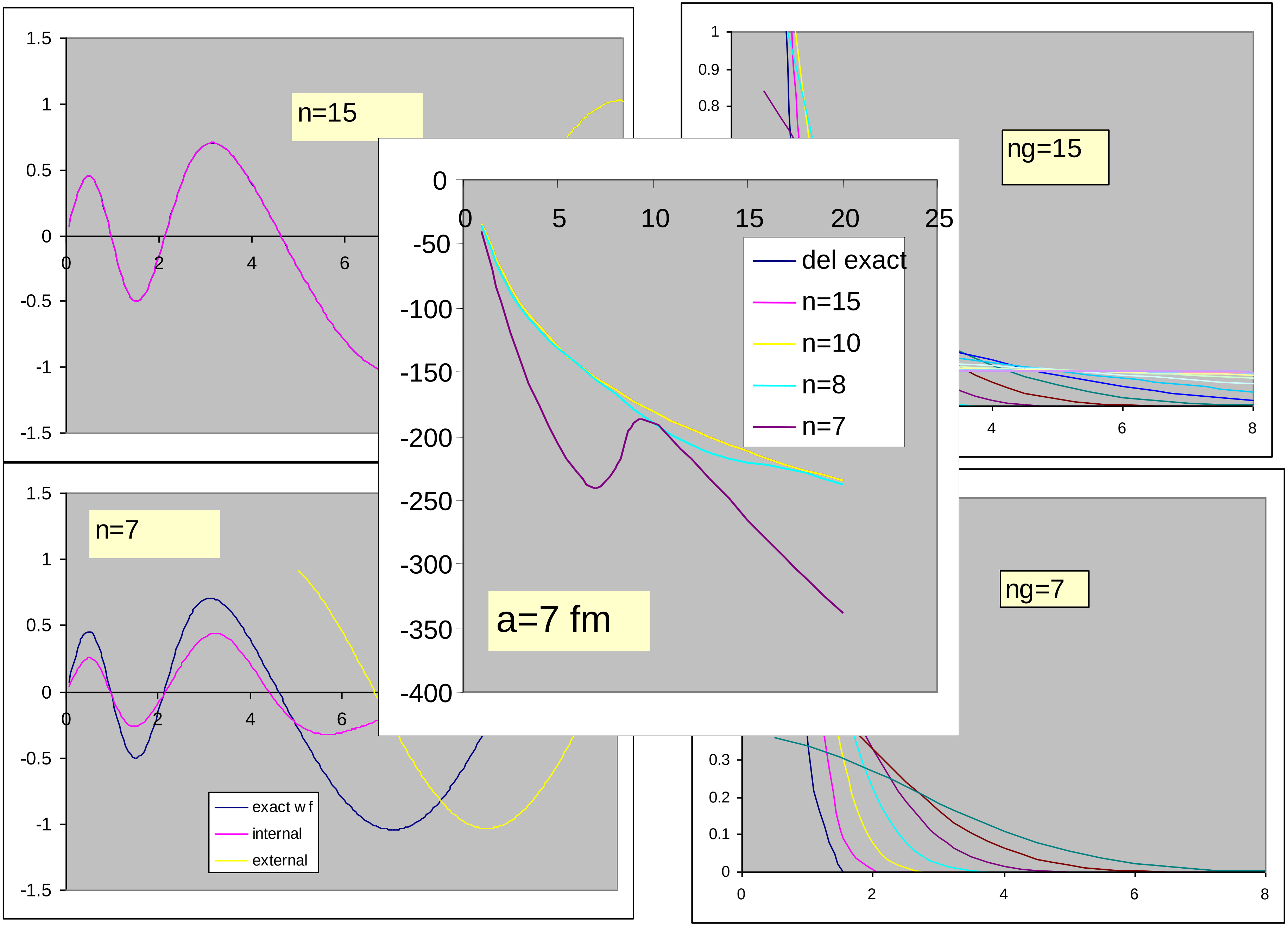
5. The calculable R-matrix Method

Elastic phase shifts



- a=10 fm too large (needs too many basis functions)
- a=4 fm too small (nuclear interaction not negligible)
- a=7 fm is a good compromise

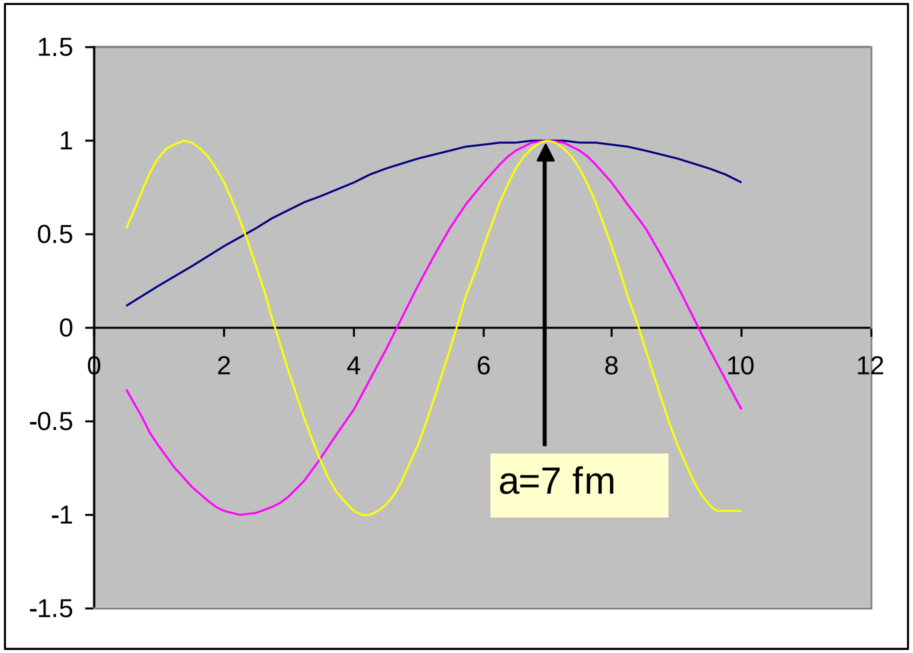
Wave functions at 5 MeV, a= 7 fm

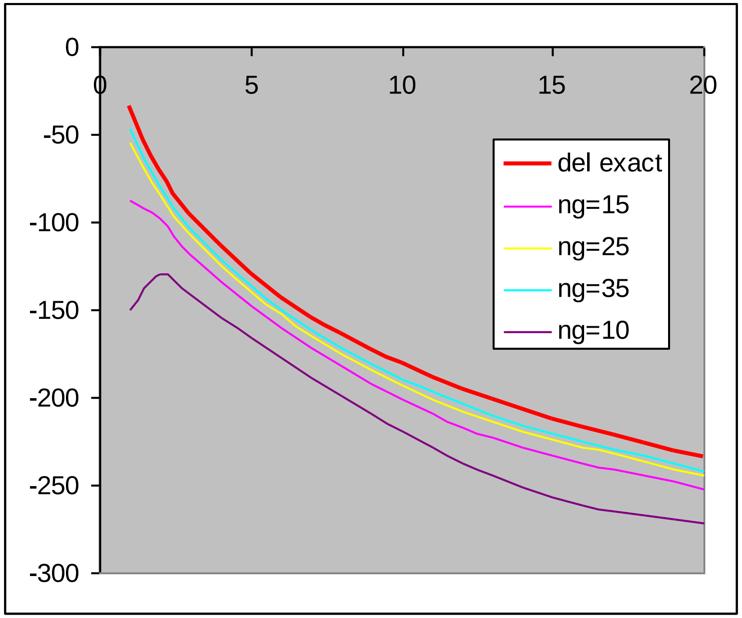


5. The calculable R-matrix Method

Other example : sine functions $\phi_i(r) = \sin\left(\frac{\pi r}{a}\left(i - \frac{1}{2}\right)\right)$

- Matrix elements very simple
- Derivative $u_i'(a)=0$

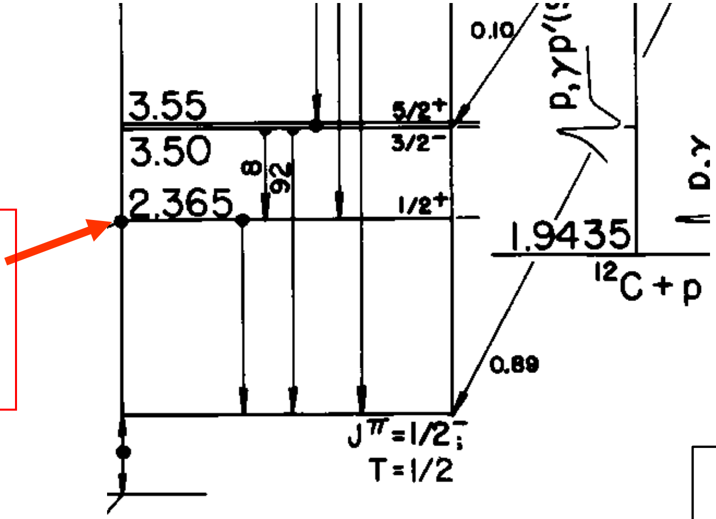




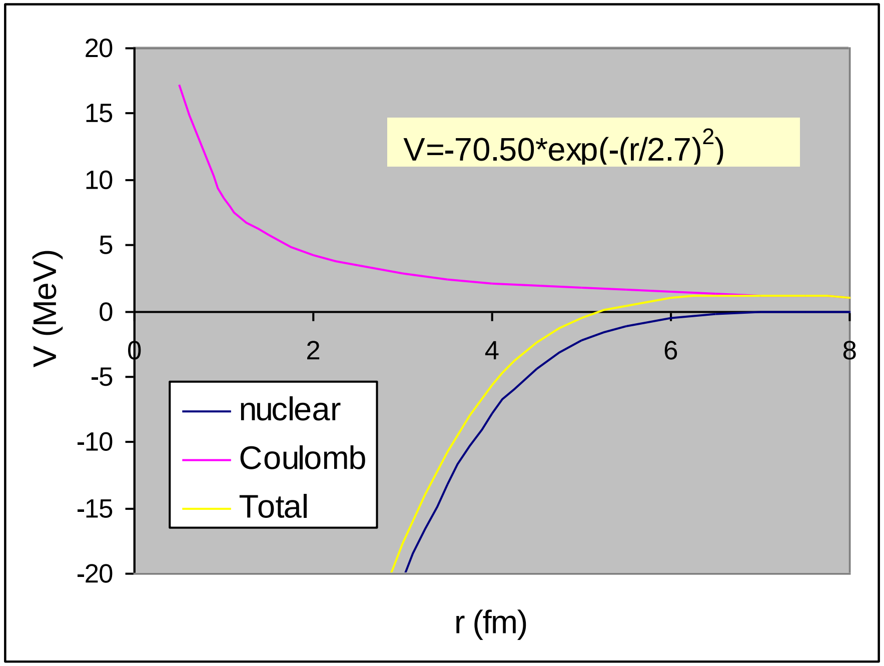
➔ Not a good basis (no flexibility)

5. The calculable R-matrix Method

Example of resonant reaction: ^{12}C+p

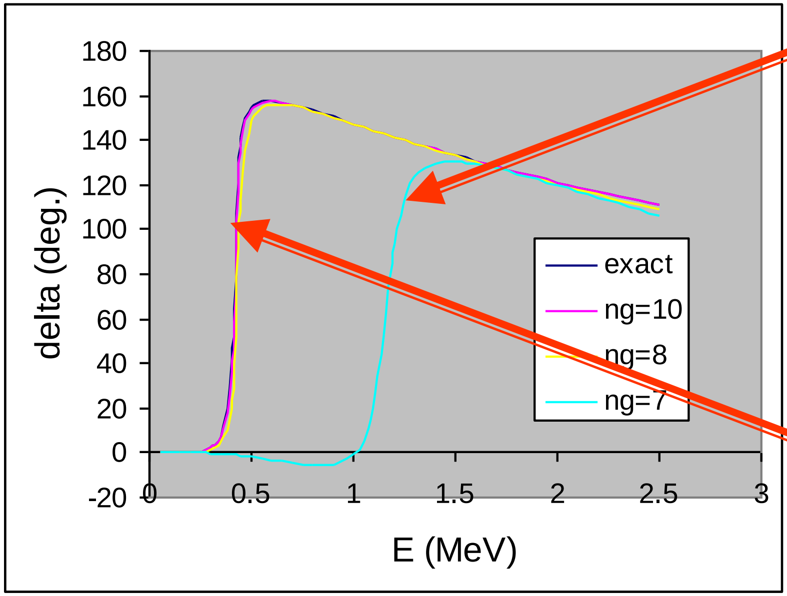


Resonance
ℓ=0
E=0.42 MeV
Γ=32 keV

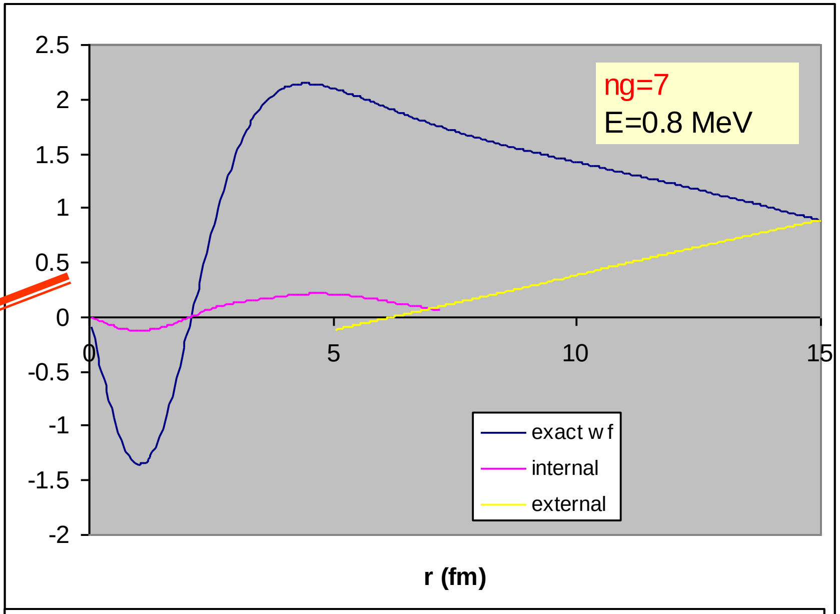


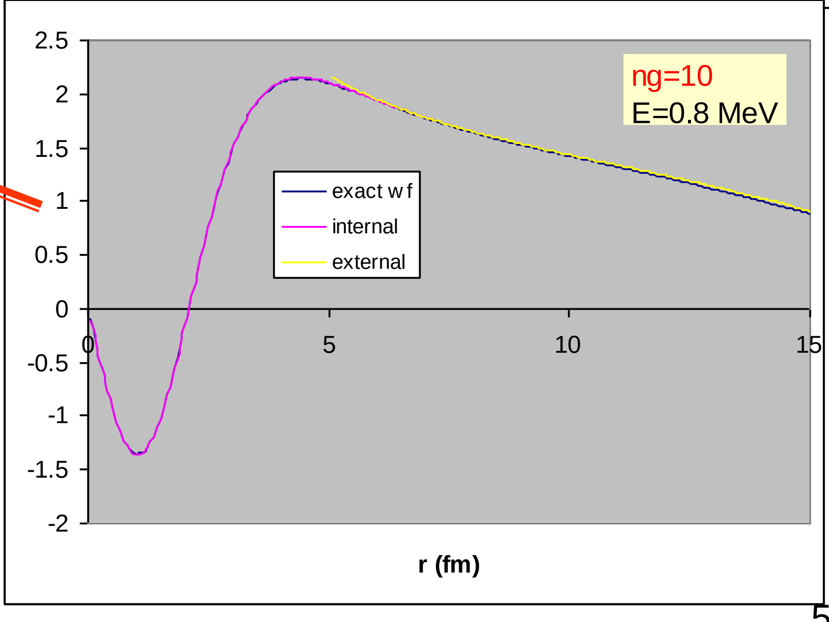
- potential : V=-70.5*exp(-(r/2.70)^2)
- Basis functions: $\phi_i(r) = r^{\ell}\exp(-(r/a_i)^2)$

Phase shifts a=7 fm



Wave functions





5. The calculable R-matrix Method

Application of the R-matrix to bound states

Positive energies: $\left(\frac{d^2}{dr^2}-\frac{\ell(\ell+1)}{r^2}-2\frac{\eta k}{r}+k^2\right)u_\ell=0$ Coulomb functions $F_\ell(kr), G_\ell(kr)$

Negative energies: $\left(\frac{d^2}{dr^2}-\frac{\ell(\ell+1)}{r^2}-2\frac{\eta k}{r}-k^2\right)u_\ell=0$ Whittaker functions $W_{-\eta,\ell+1/2}(2kr)$

Asymptotic behaviour:

$$F_\ell(x)\to\sin(x-\ell\frac{\pi}{2}-\eta\log 2x)$$

$$G_\ell(x)\to\cos(x-\ell\frac{\pi}{2}-\eta\log 2x)$$

$$W_{-\eta,\ell+1/2}(2x)\to\frac{\exp(-x)}{x^\eta}$$

5. The calculable R-matrix Method

- R matrix equations for bound states

  We use $(H - E + \mathcal{L}(L))\, u_{int} = \mathcal{L}(L) u_{ext}$ (1)

  $u_{int}(r) = \sum_{i=1}^{N} c_i \phi_i(r)$ (2)

  $u_{ext}(r) = C\, W_{-\eta,\ell+1/2}(2kr)$ (3)

With *C*=ANC (Asymptotic Normalization Constant): important in "external" processes

- Using (2) in (1) and the continuity equation:

$$\sum_{i=1}^{N} c_i < \phi_j\, |H - E + \mathcal{L}(L)|\phi_i >_{int} = < \phi_j\ (r)|\mathcal{L}(L)|u_{ext} > = 0$$

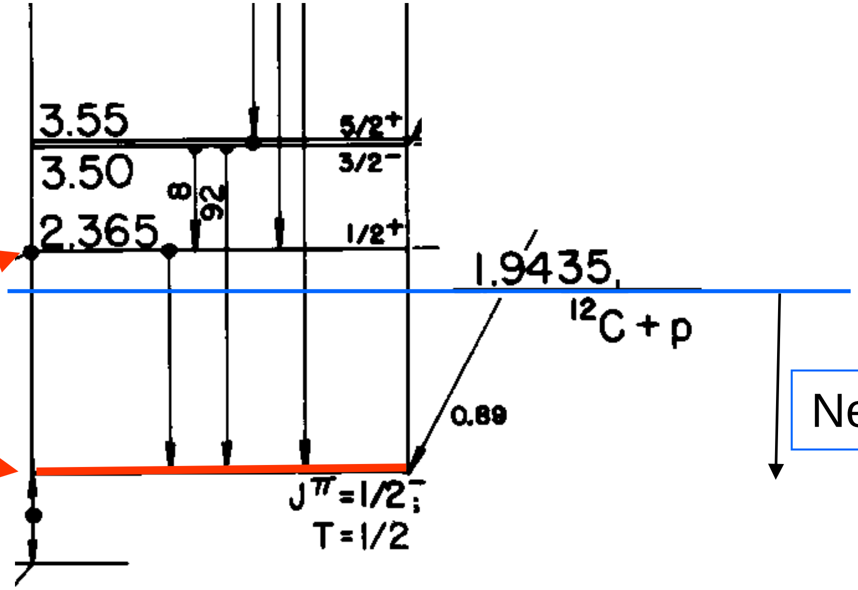
$$\text{if } L = 2ka\, W'(2ka)/W(ka)$$

- Standard diagonalization problem
- But: *L* depends on the energy, which is not known → iterative procedure

  First iteration: L=0

5. The calculable R-matrix Method

Application to the ground state of $^{13}N = ^{12}C + p$



Resonance
ℓ=0: E>0

Bound state
ℓ=1: E<0

Negative energy: -1.94 MeV

- Potential : $V=-55.3*\exp(-(r/2.70)^2)$
- Basis functions: $\phi_i(r) = r^{\ell} \exp(-(r/a_i)^2)$ (as before)

5. The calculable R-matrix Method

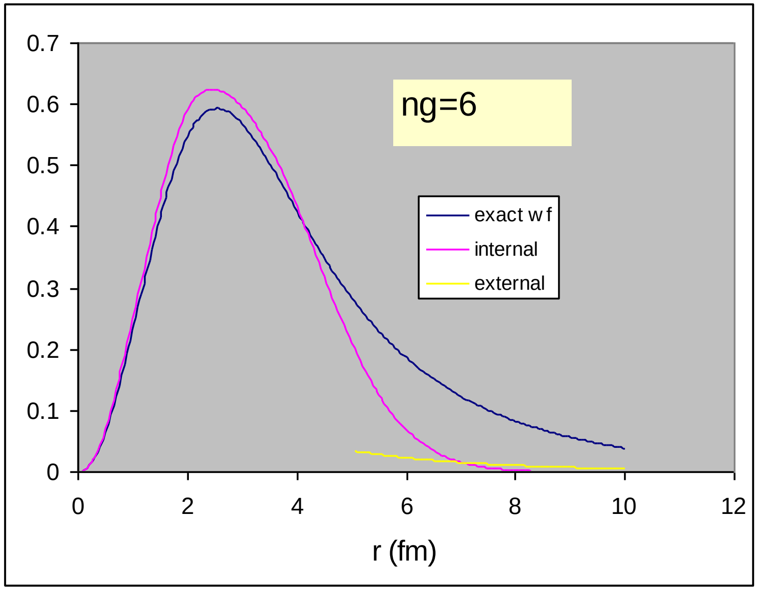
Calculation with a=7 fm

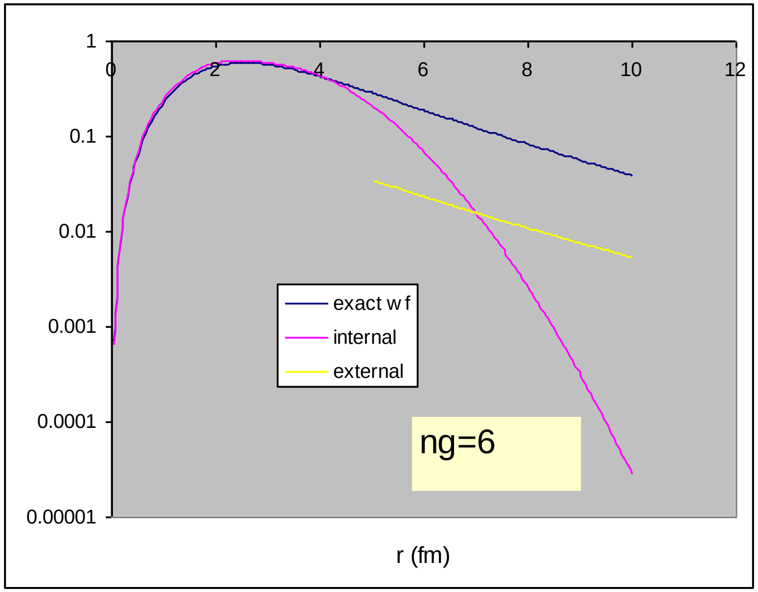
- ng=6 (poor results)
- ng=10 (good results)

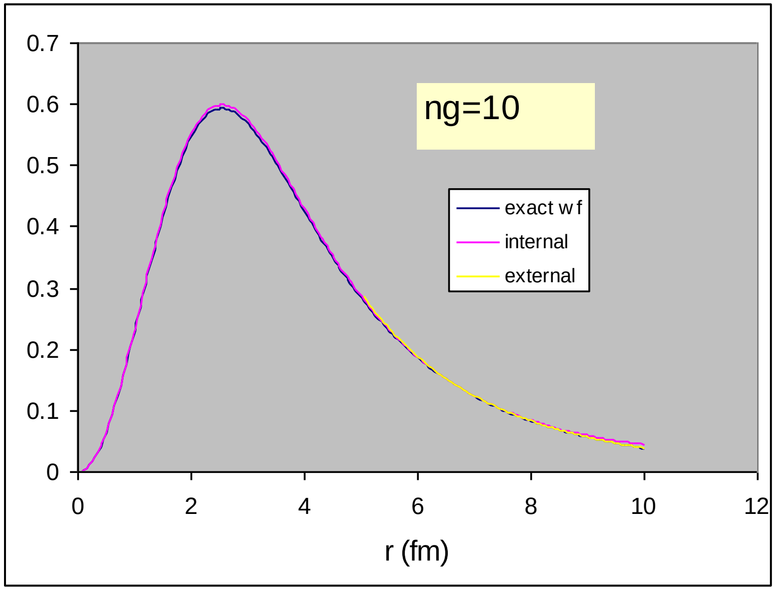
Iteration	ng=6	ng=10
1	-1.500	-2.190
2	-1.498	-1.937
3	-1.498	-1.942
		-1.942
Final	-1.498	-1.942
Exact	-1.942	
Left derivative	-1.644	-0.405
Right derivative	-0.379	-0.406

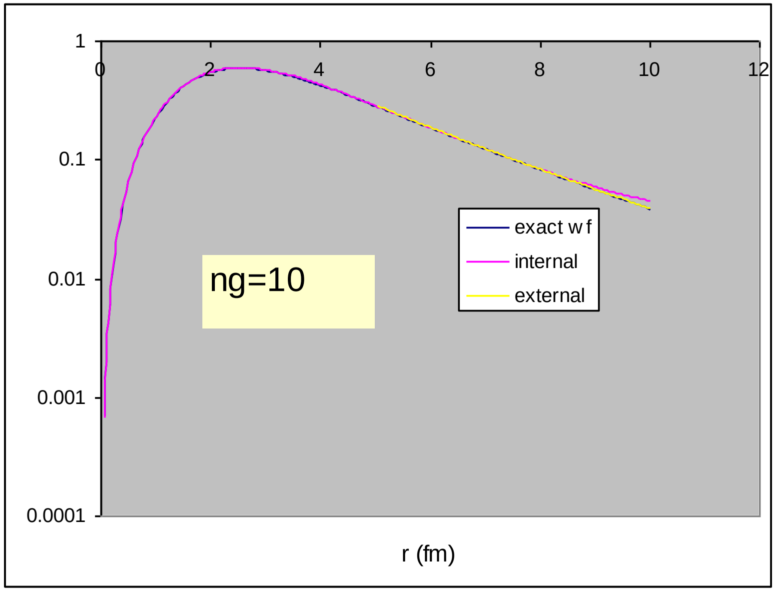
5. The calculable R-matrix Method

Wave functions (a=7 fm)









5. The calculable R-matrix Method

Other applications of the “calculable” R matrix

- **Microscopic cluster models**:
  include the structure of the nuclei
  use a nucleon-nucleon interaction

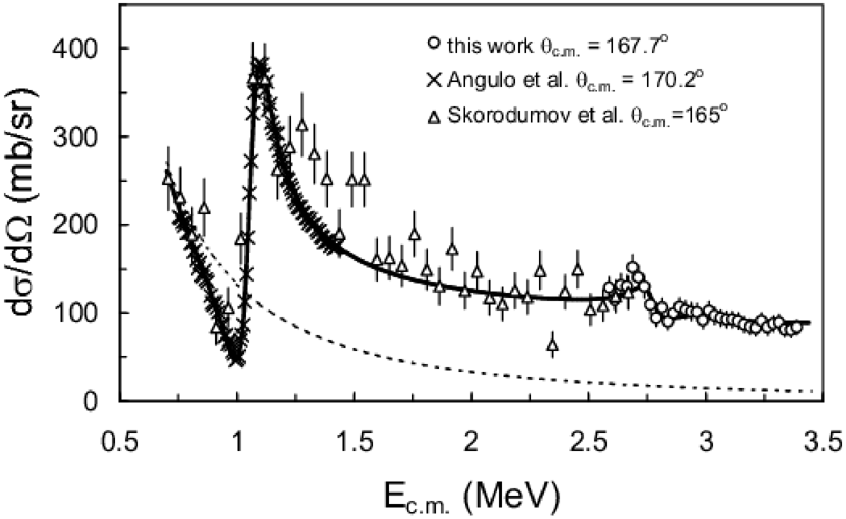
- **Three-body continuum states**: P.D., E. Tursunov, D. Baye, NPA765 (2006) 370
  hyperspherical formalism (ex: α+n+n)
  set of coupled equations

- **CDCC calculations**
  Reactions with 2-body projectiles (ex: d+^{58}Ni) or 3-body projectiles (^{6}He)
  set of coupled equations

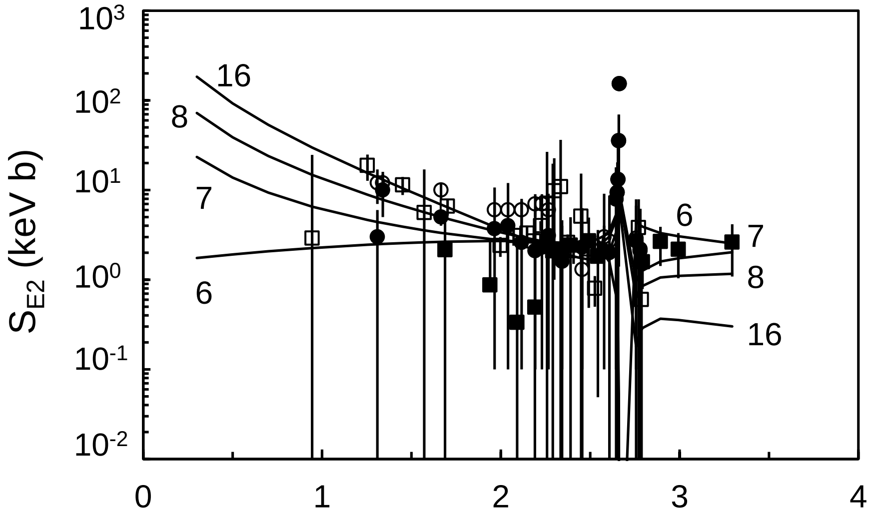
- **Many applications in atomic physics**

6. Phenomenological R matrix Method

- Main Goal: fit of experimental data



^{18}Ne+p elastic scattering
→ resonance properties



Nuclear astrophysics: $^{12}C(\alpha,\gamma)^{16}O$
→Extrapolation to low energies

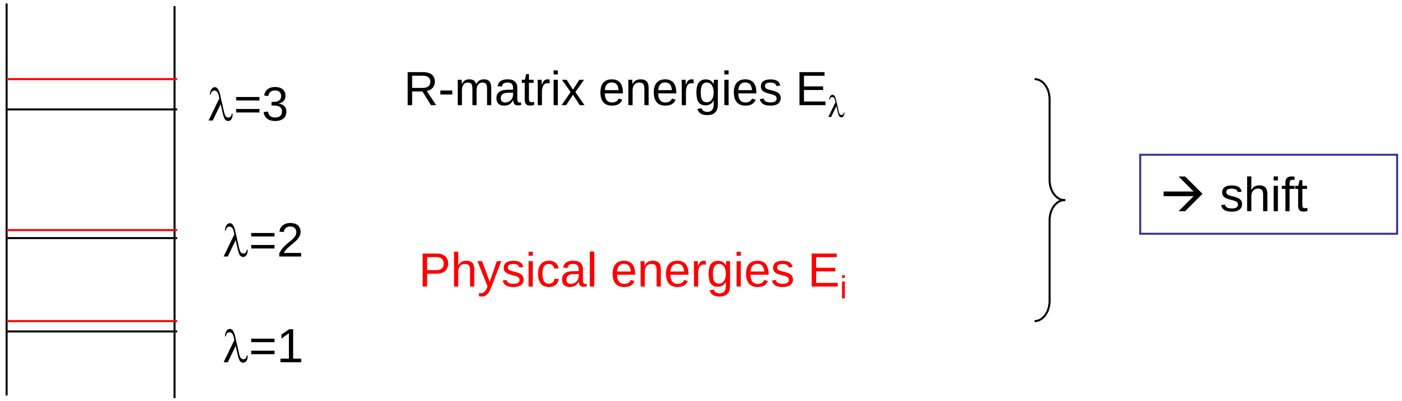
6. Phenomenological R matrix Method

General definition of the R-matrix: $R(E) = \sum_{\lambda=1}^{N} \frac{\gamma_\lambda^2}{E_\lambda - E}$

- Calculable R-matrix: parameters $E_\lambda, \gamma_\lambda^2$ are **calculated from basis functions**
- Phenomenological R-matrix: parameters are **fitted to data** (phase shifts, cross sections, etc.)
- Must be done for each ℓ value → adapted to low level densities
- In general: single-pole approximation $R(E) \approx \frac{\gamma_0^2}{E_0 - E}$

Main problem: what is the link between

- The R matrix parameters (="formal" parameters)
- The physical parameters (="observed" parameters)



R-matrix energies E_λ

Physical energies E_i

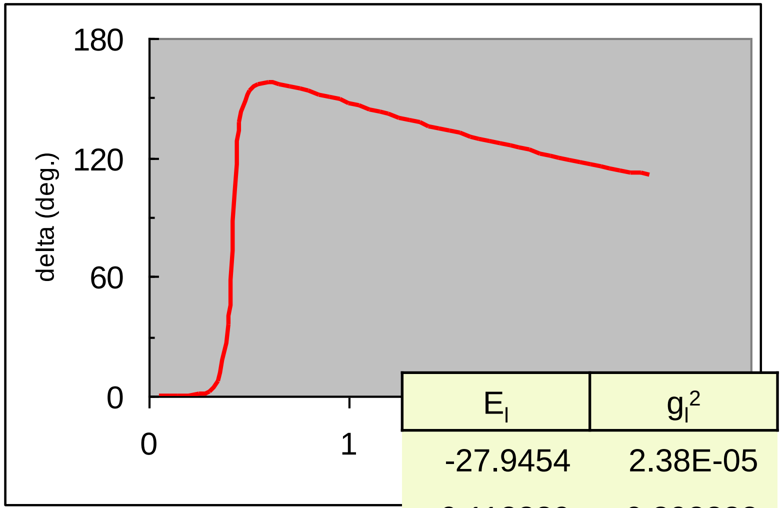
→ shift

6. Phenomenological R matrix Method

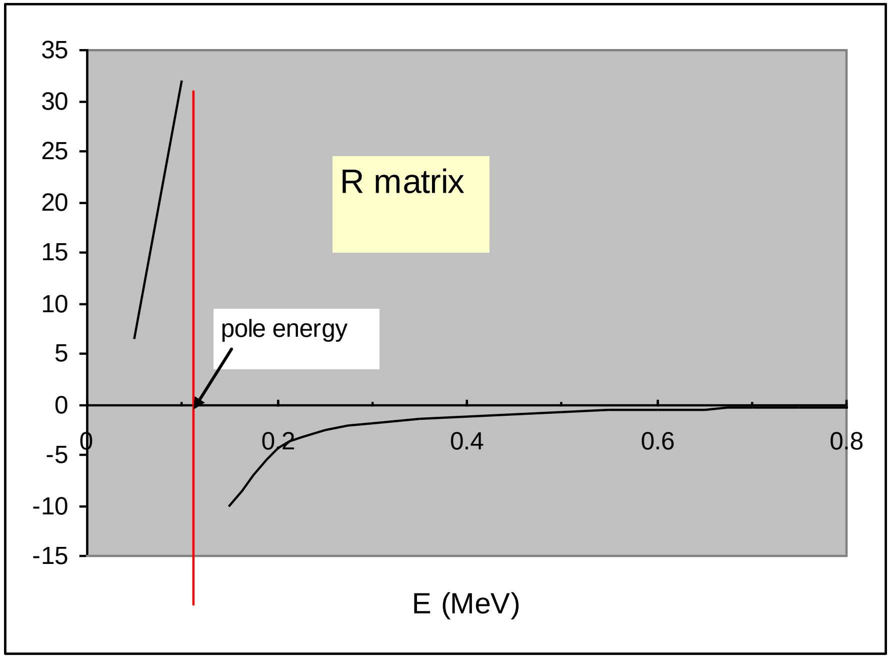
Example : ^{12}C+p

- potential : $V=-70.5*\exp(-(r/2.70)^2)$
- Basis functions: $u_i(r)=r^{\ell}*\exp(-(r/a_i)^2)$ with $a_i=x_0*a_0^{(i-1)}$

10 basis functions, a=8 fm



$$R(E)=\sum_{\lambda}\frac{\gamma_{\lambda}^{2}}{E_{\lambda}-E}$$



10 eigenvalues

E_l	g_l^2
-27.9454	2.38E-05
0.112339	0.392282
7.873401	1.094913
26.50339	0.730852
40.54732	0.010541
65.62125	1.121593
107.7871	4.157215
153.8294	1.143461
295.231	0.097637
629.6709	0.019883

6. Phenomenological R matrix Method

Calculation: 10 poles

pole	E_l	γ_l^2
1	-27.95	2.38E-05
2	0.11	3.92E-01
3	7.87	1.09E+00
4	26.50	7.31E-01
5	40.55	1.05E-02
6	65.62	1.12E+00
7	107.79	4.16E+00
8	153.83	1.14E+00
9	295.23	9.76E-02
10	629.67	1.99E-02

Fit to data

Isolated pole (2 parameters)

$$\frac{\gamma_0^2}{E_0 - E}$$

Background (high energy): gathered in 1 term

$$R_0(E) = \sum_{\lambda \neq 0} \frac{\gamma_\lambda^2}{E_\lambda - E}$$

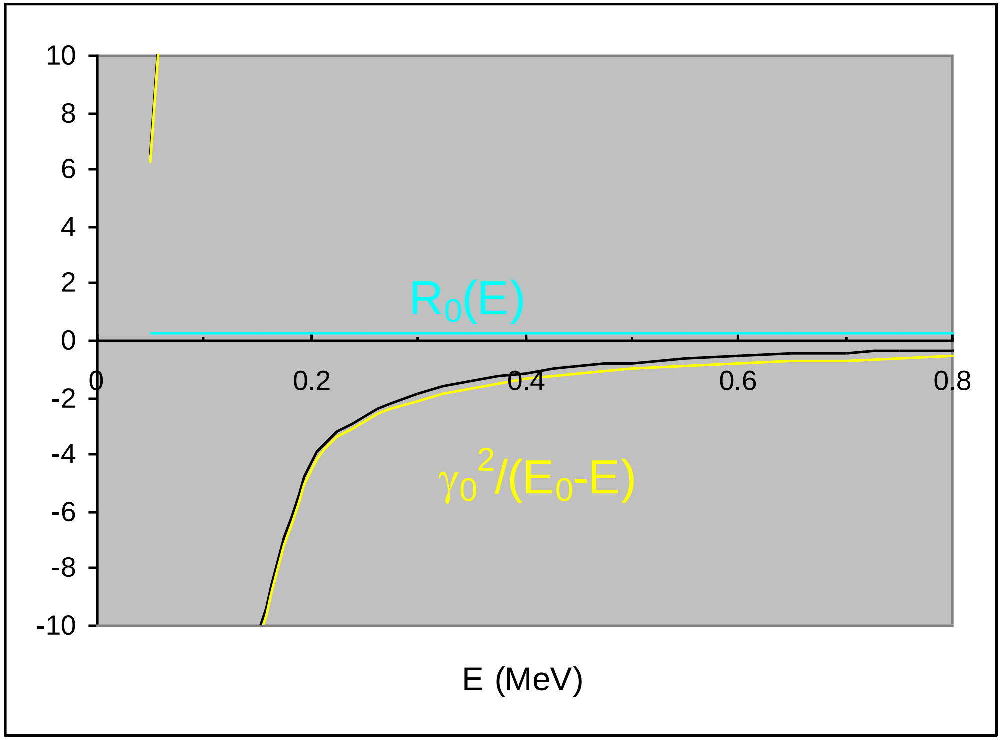
$E \ll E_l \rightarrow R_0(E) \sim R_0$

➔ In phenomenological approaches (one resonance): $R(E) \approx \frac{\gamma_0^2}{E_0 - E} + R_0$

6. Phenomenological R matrix Method

^{12}C+p

$$R(E) = \sum_{\lambda} \frac{\gamma_\lambda^2}{E_\lambda - E} = \frac{\gamma_0^2}{E_0 - E} + R_0(E)$$



Approximations: $R_0(E)=R_0$=constant (background)

$R_0(E)=0$: Breit-Wigner approximation: one term in the R matrix

Remark: the R matrix method is NOT limited to resonances ($R=R_0$)

6. Phenomenological R matrix Method

Resonance energies

Relation between the collision matrix and the R matrix

$$\begin{aligned} U^\ell &= \frac{I_\ell(ka)}{O_\ell(ka)} \frac{1 - L^\star R^\ell}{1 - L R^\ell}, \text{ with } L(E) = S(E) + iP(E) \\ &= \exp(2i\delta^\ell) = \exp(2i(\delta^\ell_{HS} + \delta^\ell_R)) \end{aligned}$$

with

$$\exp(2i\delta^\ell_{HS}) = \frac{I_\ell(ka)}{O_\ell(ka)} \rightarrow \delta^\ell_{HS} = -\arctan \frac{F_\ell(ka)}{G_\ell(ka)}$$

Hard-sphere

$$\exp(2i\delta^\ell_R) = \frac{1 - L^\star R^\ell}{1 - L R^\ell} \rightarrow \delta^\ell_R = \arctan \frac{PR}{1 - SR}$$

R-matrix

Resonance energy E_r defined by $1 - S(E_r)R(E_r) = 0$ → $\delta_R = 90^o$

In general: must be solved numerically

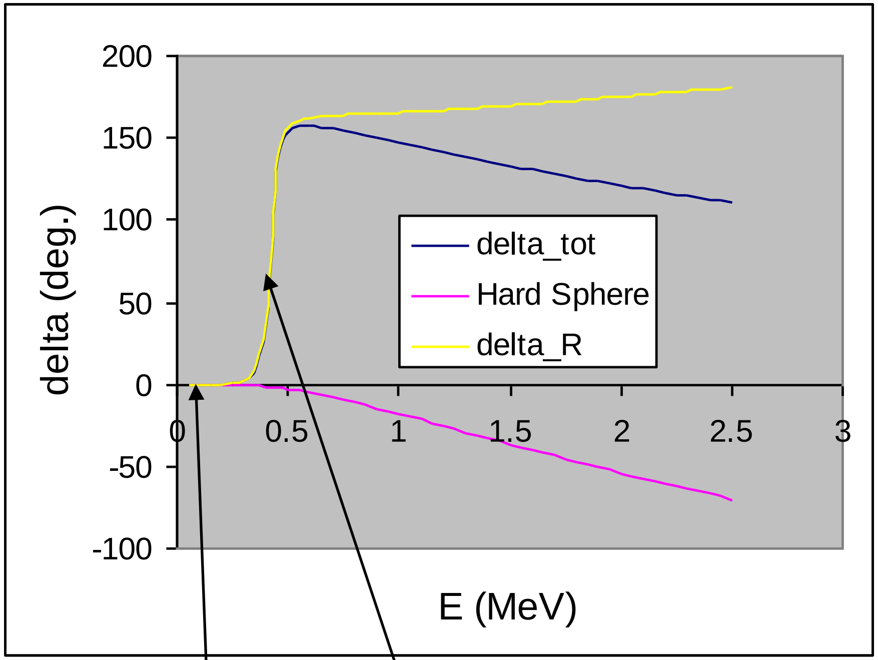
6. Phenomenological R matrix Method

$$\delta_R^{\ell}(E) = \arctan \frac{P(E)R(E)}{1 - S(E)R(E)}$$

Resonance energy E_r defined by $1 - S(E_r)R(E_r) = 0$ →δ_R=90°

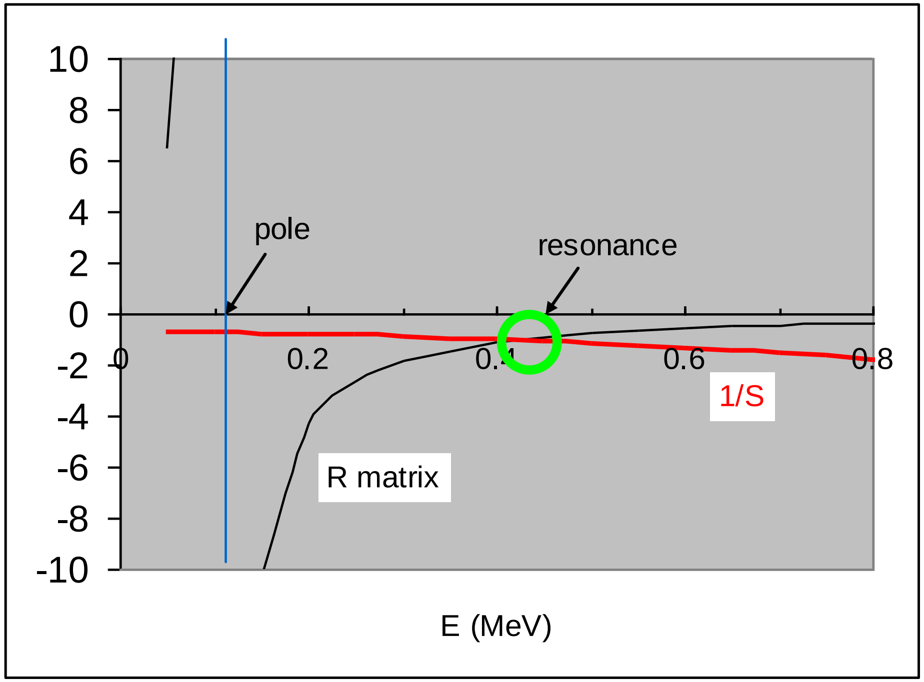
In general: must be solved numerically

Plot of R(E), 1/S(E)



pole

resonance

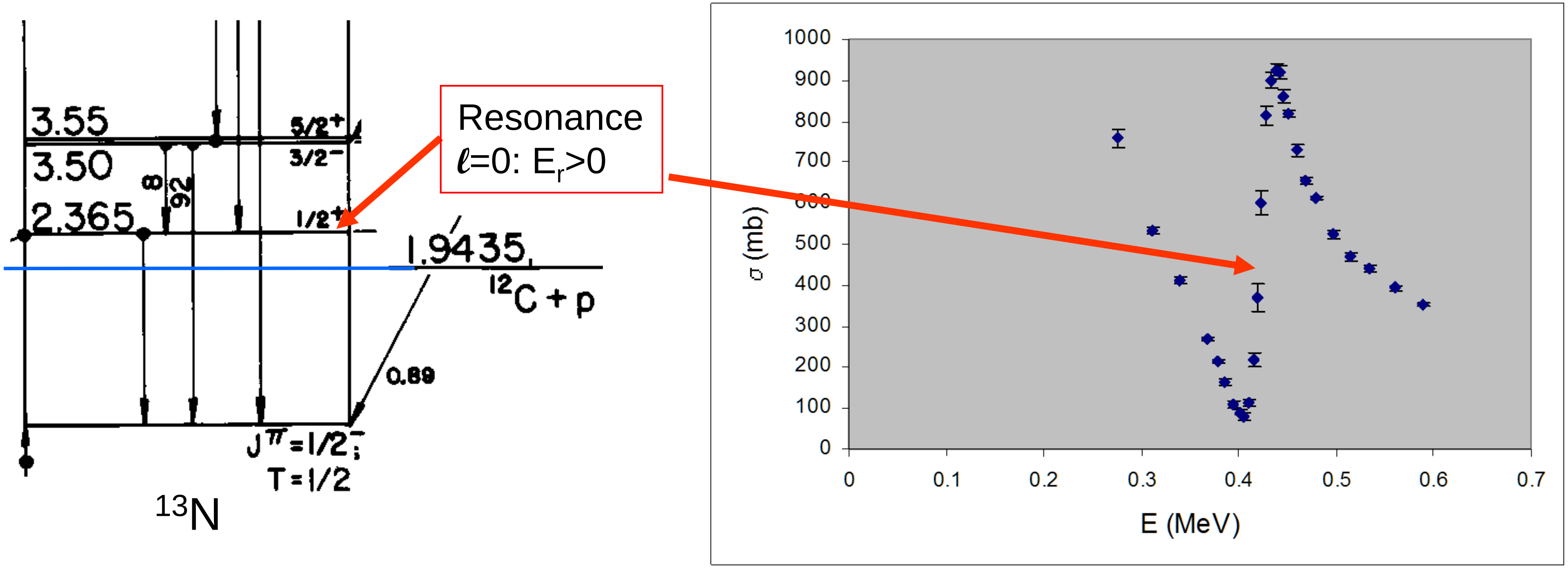


6. Phenomenological R matrix Method

Simple case: one pole (="isolated" resonance):

$$R(E) = \frac{\gamma_0^2}{E_0 - E}$$

Example: ^{12}C+p: E_R=0.42 MeV



Resonance
ℓ=0: E_r>0

Question: link between E_0 and E_R? ➔ Breit-Wigner

6. Phenomenological R matrix Method

The Breit-Wigner approximation

Single pole in the R matrix expansion: $R(E) = \frac{\gamma_0^2}{E_0 - E}$

Phase shift: $$\tan \delta_R(E) = \frac{P(E)R(E)}{1 - S(E)R(E)} \approx \frac{\gamma_0^2 P(E)}{E_0 - E - \gamma_0^2 S(E)}$$
$$\approx \frac{\Gamma(E)}{2(E_r - E)}$$

Thomas approximation: $S(E) \approx S(E_0) + S'(E_0)(E - E_0)$

Then: $$E_r \approx E_0 - \frac{\gamma_0^2 S(E_0)}{1 + \gamma_0^2 S'(E_0)}$$
$$\Gamma(E) = 2\frac{\gamma_0^2}{1 + \gamma_0^2 S'(E_r)} P(E) = 2\gamma_{obs}^2 P(E)$$

At the resonance: $\Gamma_r = \Gamma(E_r) = 2\gamma_{obs}^2 P(E_r)$: strongly depends on energy

➔ Breit-Wigner = R-matrix with one pole

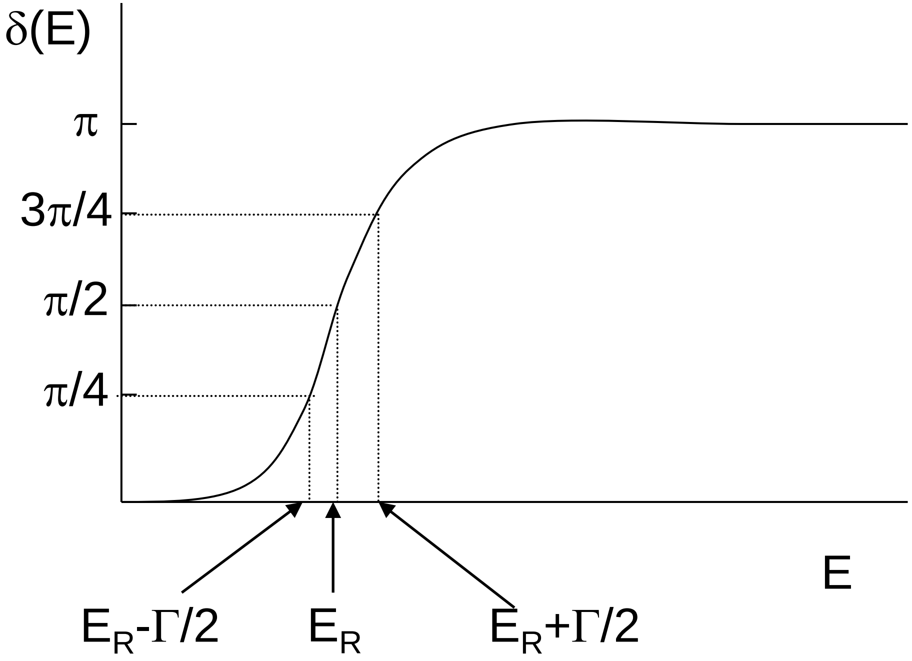
➔ Generalization possible (more than one pole: interference effects)

6. Phenomenological R matrix Method

Phase shift for one pole

Breit-Wigner approximation: $\delta_R(E) \approx \frac{\Gamma}{2(E_R - E)}$ =one-pole R matrix

E_R=resonance energy
Γ=resonance width



- Narrow resonance: Γ small
- Broad resonance: Γ large

6. Phenomenological R matrix Method

Link between “calculated” and “observed” parameters

One pole (N=1)

$$E_R = E_0 - \frac{S(E_0)\gamma_0^2}{1 + S'(E_0)\gamma_0^2}$$

$$\gamma_{obs}^2 = \frac{\gamma_0^2}{1 + S'(E_0)\gamma_0^2}$$

R-matrix parameters
(calculated)

↕

Observed parameters
(=data)

Several poles (N>1)

$$1 - S(E_r)R(E_r) = 0$$

Must be solved numerically

Generalization of the Breit-Wigner formalism:
link between observed and formal parameters when *N*>1

C. Angulo, P.D., Phys. Rev. C **61**, 064611 (2000) – single channel

C. Brune, Phys. Rev. C **66**, 044611 (2002) – multi channel

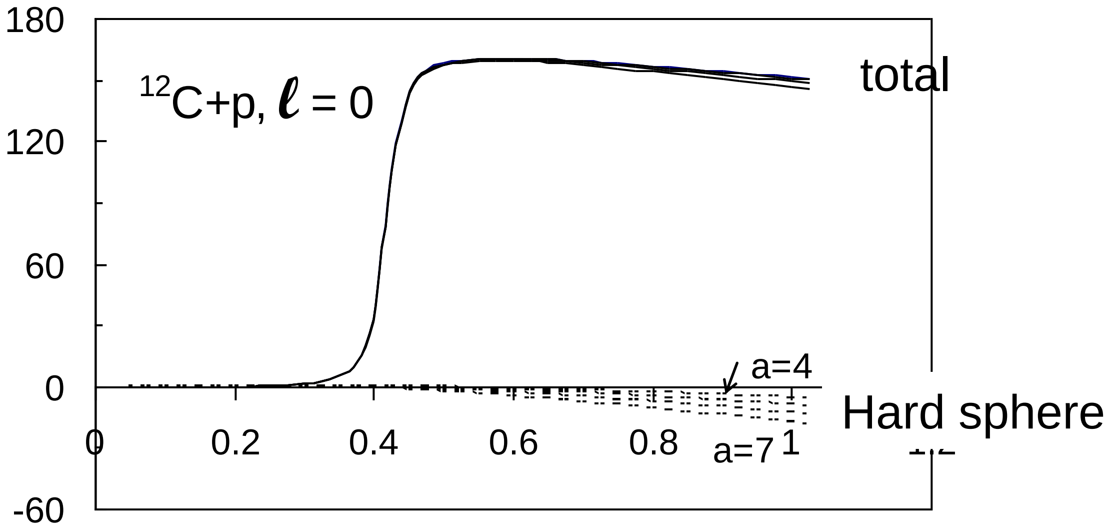
6. Phenomenological R matrix Method

Examples: ^{12}C+p and ^{12}C+α

Narrow resonance: ^{12}C+p

^{12}C+p ($E^r = 0.42$ MeV, $\Gamma = 32$ keV, $J = 1/2^+, \ell = 0$)

	$a = 4$ fm	$a = 5$ fm	$a = 6$ fm	$a = 7$ fm
γ^2_{obs} (MeV)	1.09	0.59	0.35	0.23
E_0 (MeV)	−2.15	−0.61	−0.11	0.11
γ^2_0 (MeV)	3.09	1.16	0.57	0.32

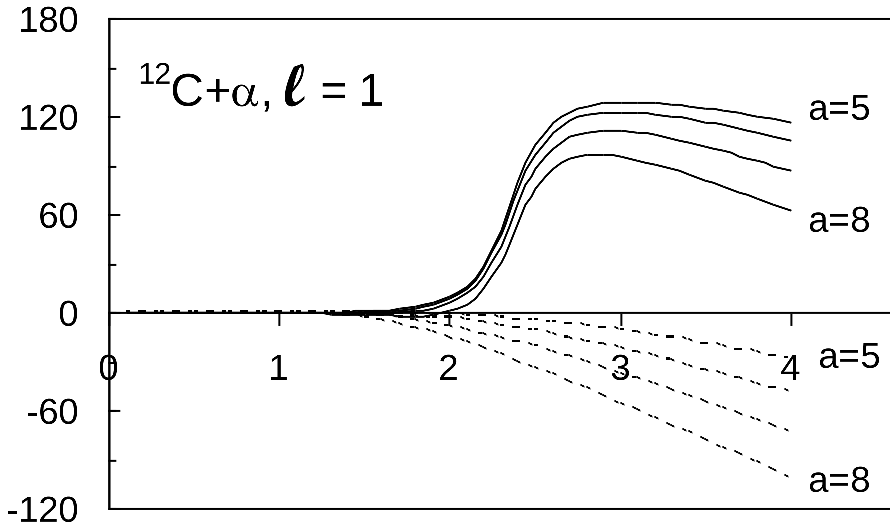


6. Phenomenological R matrix Method

Broad resonance: ^{12}C+α

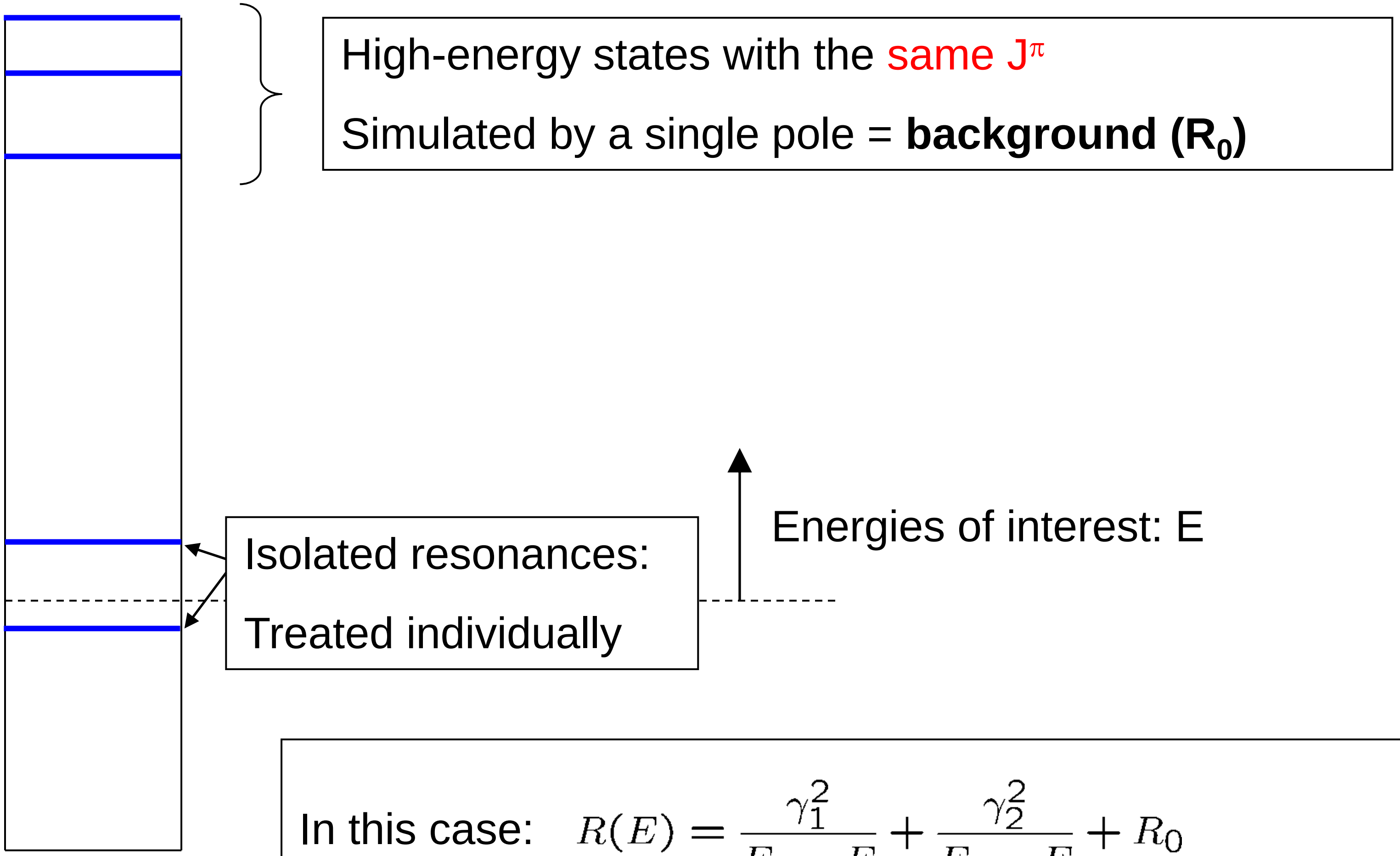
^{12}C$+\alpha$ ($E^r = 2.42$ MeV, $\Gamma = 0.42$ MeV, $J = 1^-, \ell = 1$)

	$a = 5$ fm	$a = 6$ fm	$a = 7$ fm
γ^2_{obs} (MeV)	0.57	0.28	0.16
E_0 (MeV)	0.49	1.92	2.22
γ^2_0 (MeV)	1.17	0.37	0.19



6. Phenomenological R matrix Method

Extension to multi-resonances



High-energy states with the same J^π

Simulated by a single pole = **background (R_0)**

Isolated resonances:

Treated individually

Energies of interest: E

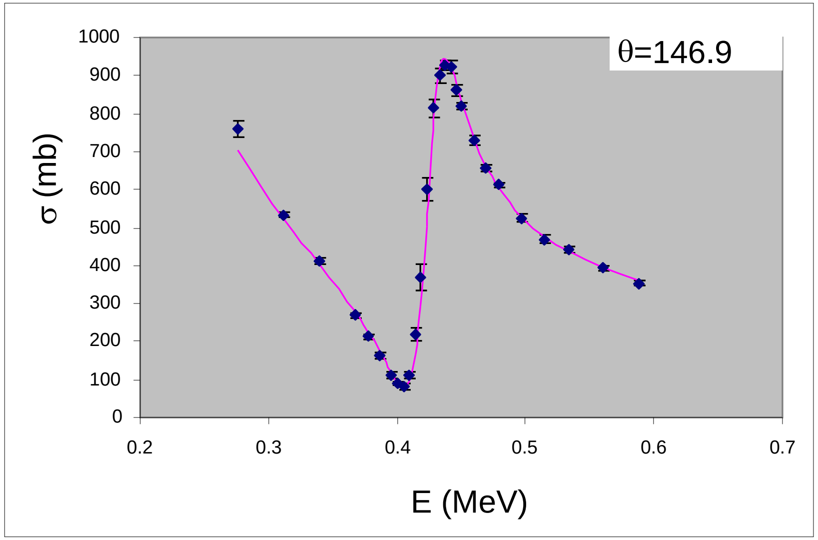
In this case: $R(E) = \frac{\gamma_1^2}{E_1 - E} + \frac{\gamma_2^2}{E_2 - E} + R_0$

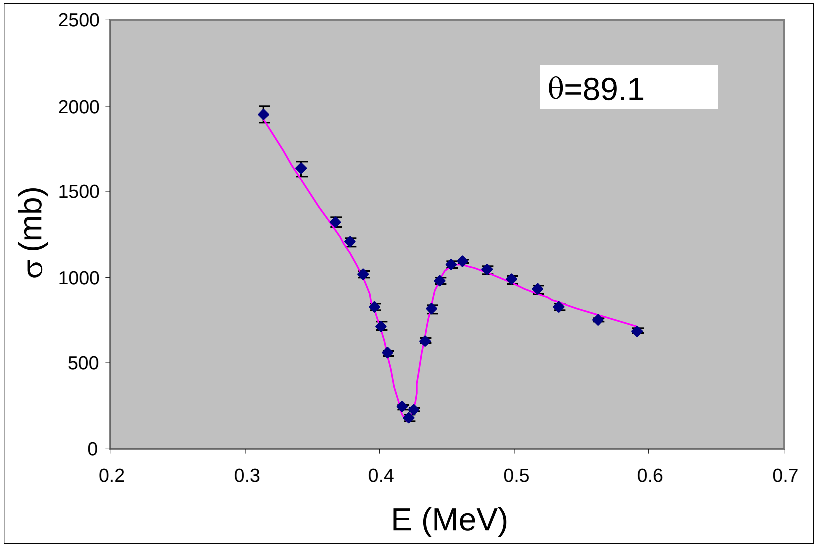
Non resonant calculations possible: $R(E) = R_0$

6. Phenomenological R matrix Method

First example: Elastic scattering $^{12}C+p$

Data from H.O. Meyer et al., Z. Phys. A279 (1976) 41





R matrix fits for different channel radii

a	E_R	Γ	E_0	$\gamma_0 2$	χ^2
4.5	0.4273	0.0341	-1.108	1.334	2.338
5	0.4272	0.0340	-0.586	1.068	2.325
5.5	0.4272	0.0338	-0.279	0.882	2.321
6	0.4271	0.0336	-0.085	0.745	2.346

➔ E_R, Γ very stable with a

➔ global fit independent of a

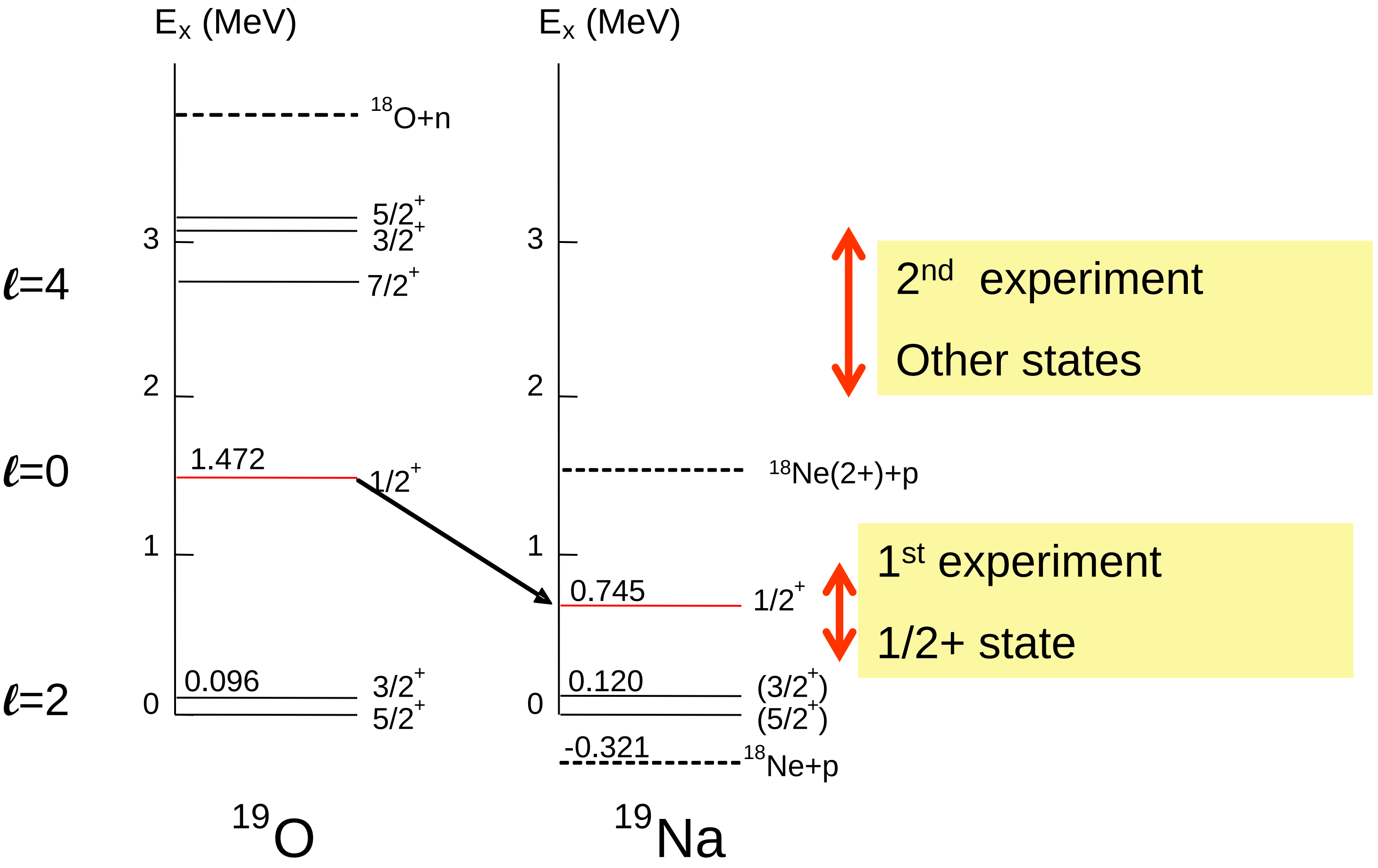
6. Phenomenological R matrix Method

Second example: ^{18}Ne+p scattering at Louvain-la-Neuve

First Experiment : ^{18}Ne+p elastic: *C. Angulo et al, Phys. Rev. C67 (2003) 014308*

→ search for the mirror state of ^{19}O(1/2$^+$)

Second experiment: ^{18}Ne(p,p')^{18}Ne(2$^+$): *M.G. Pellegriti et al, PLB 659 (2008) 864*



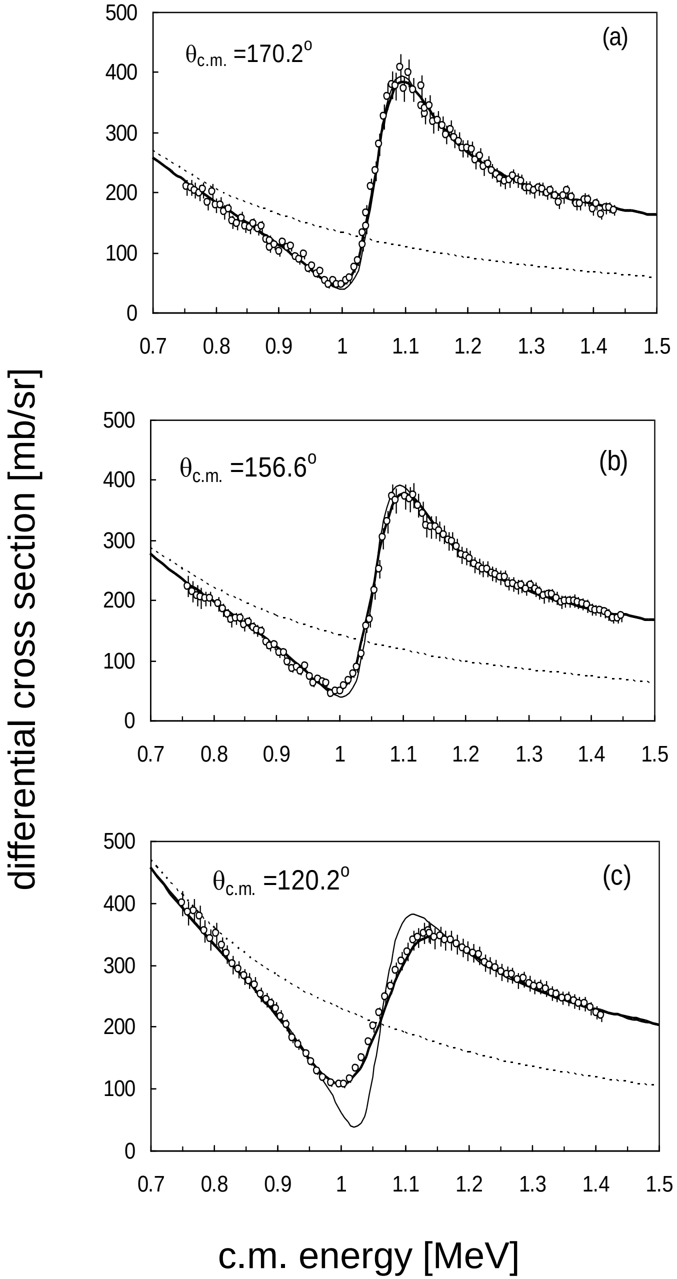
2nd experiment

Other states

1st experiment

1/2+ state

6. Phenomenological R matrix Method

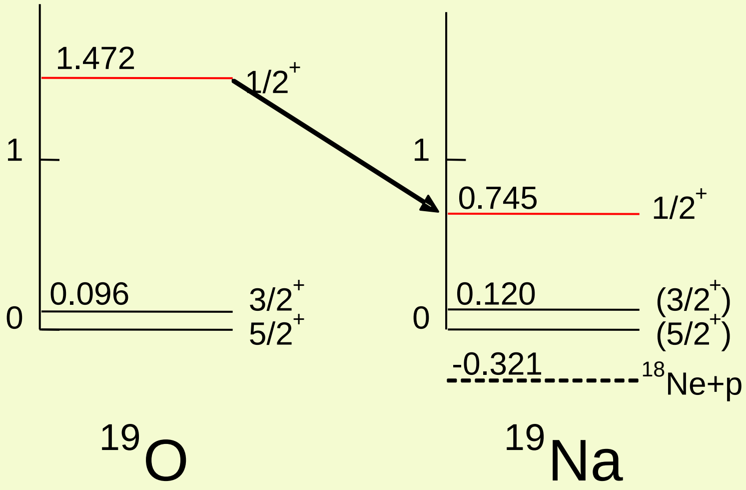


^{18}Ne+p elastic scattering

Final result

E_R = 1.066 ± 0.003 MeV

Γ_p = 101 ± 3 keV



→ Very large Coulomb shift
From Γ=101 keV, γ^2=605 keV, θ^2=23%
Very large reduced width
="single-particle state"

6. Phenomenological R matrix Method

3rd example: ^{18}Ne(p,p')^{18}Ne(2^+) inelastic scattering

Combination of ^{18}Ne(p,p)^{18}Ne elastic and ^{18}Ne(p,p')^{18}Ne(2^+) inelastic

➔ constraints on the R-matrix parameters

Generalization to 2 channels: $R_{ij}(E) = \frac{\gamma_i \gamma_j}{E_0 - E}$

i=1: ^{18}Ne(0^+)+p channel

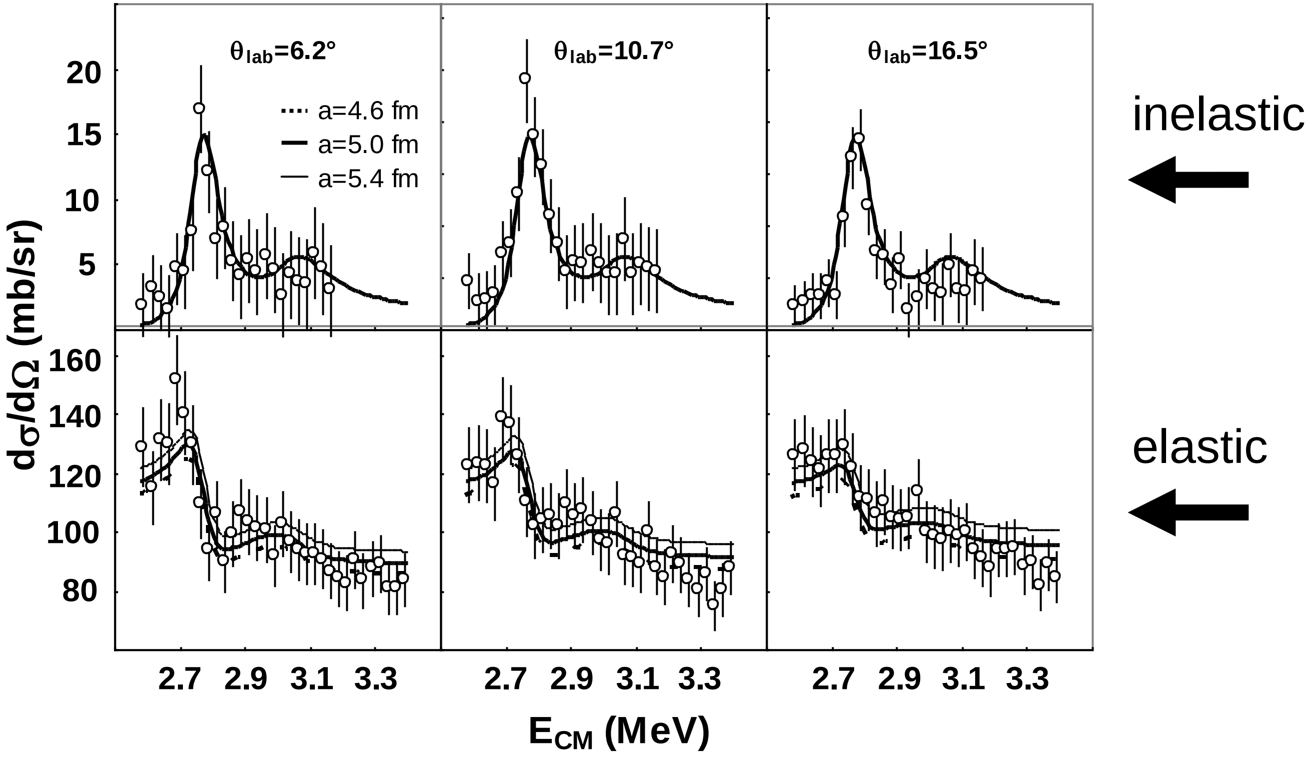
i=2: ^{18}Ne(2^+)+p channel

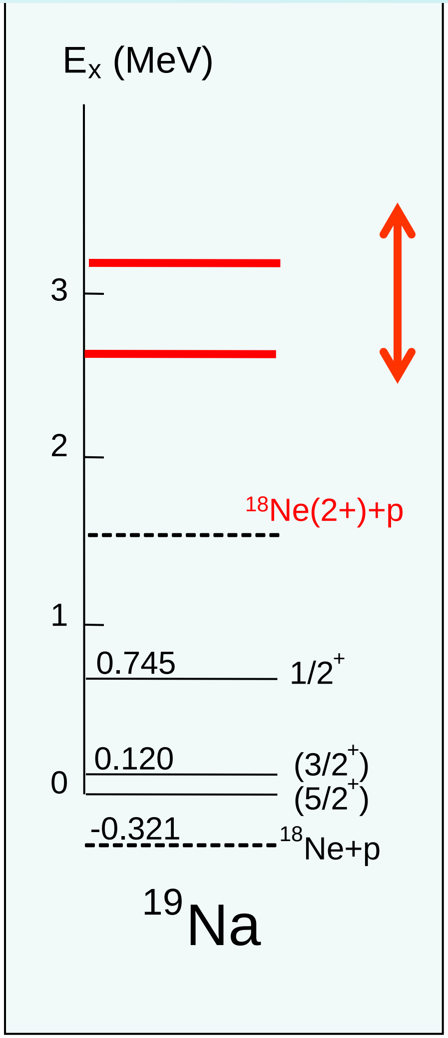
➔ each state has 3 parameters: E_0, γ_1, γ_2

From R-matrix → collision matrix U_{ij}
Elastic cross section obtained from U_{11}
Inelastic cross section obtained from U_{12}

6. Phenomenological R matrix Method

^{18}Ne+p inelastic scattering: *M.G. Pellegriti et al, PLB 659 (2008) 864*





Several angles fitted simultaneously

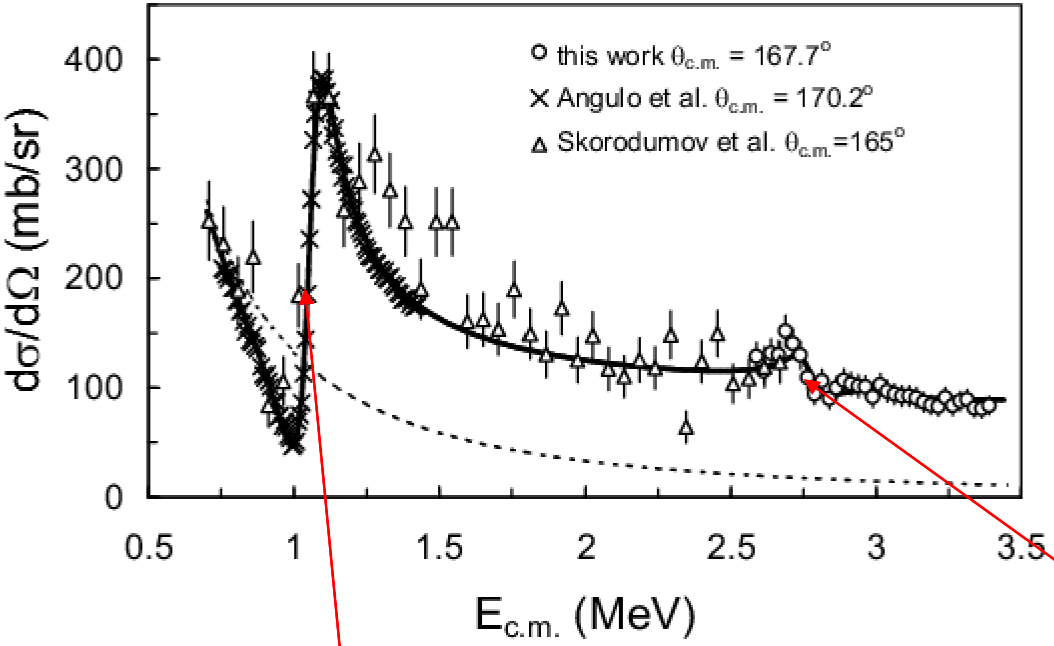
Presence of 2 states → 6 parameters

$E_{\mathrm{c.m.}}$ (MeV)	$2J^\pi$	Γ_{tot} (keV)	$(2J+1)\frac{\Gamma_0}{\Gamma_{\mathrm{tot}}}$	θ_0^2 (%)	θ_2^2 (%)
2.78 ± 0.03	$(5,3)^+$	105 ± 10	0.43 ± 0.05	1.1 ± 0.3	44 ± 4
3.09 ± 0.06	$(3,5)^+$	250 ± 50	0.12 ± 0.04	0.6 ± 0.2	36 ± 7

→dominant
p+^{18}Ne(2^+) structure

6. Phenomenological R matrix Method

^{18}Ne+p elastic scattering: comparison with other experiments



Complete set of data up to 3.5 MeV

θ_0 large ~23%

→ observable in elastic scattering

→ dominant p+^{18}Ne(0^+) structure

→ single-particle state

θ_2 large (2 states) ~40%, θ_0 small

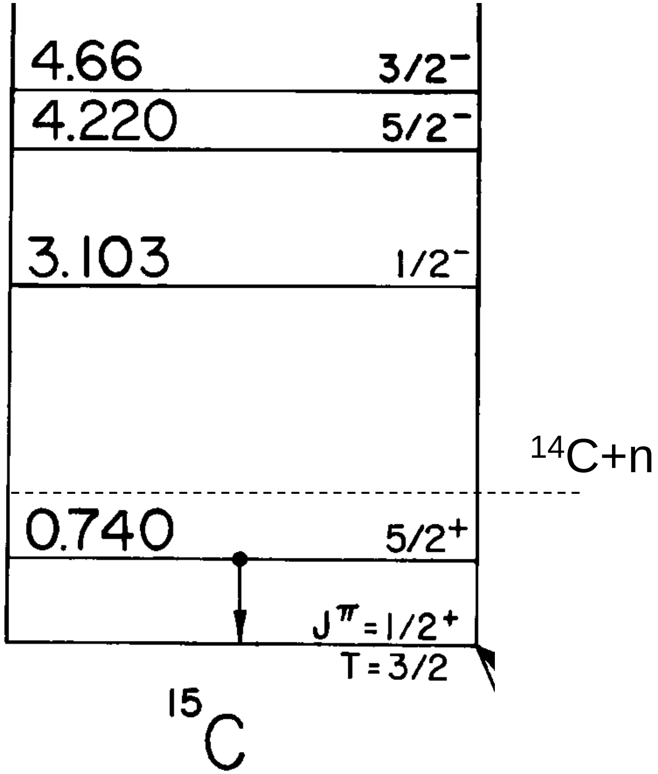
→ difficult to observe in elastic scattering

→ dominant p+^{18}Ne(2^+) structure

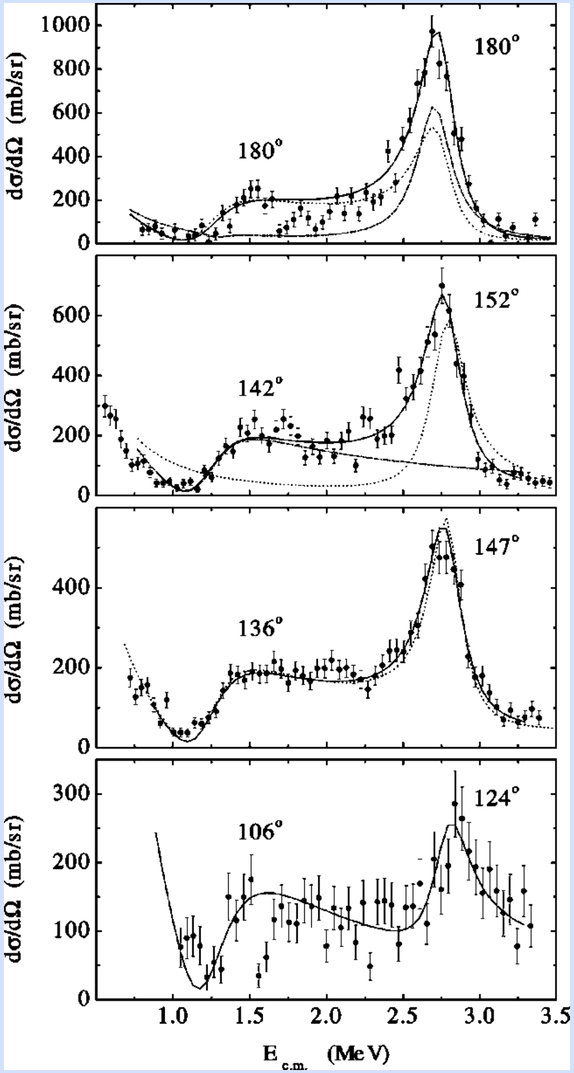
→ single-particle states with excited core

6. Phenomenological R matrix Method

Other example: $^{14}O+p \rightarrow {}^{15}F$: analog of ^{15}C

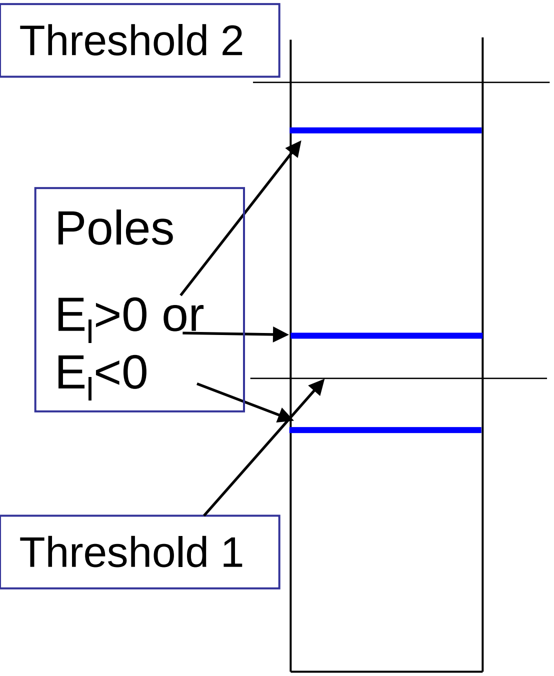


^{14}O+p elastic: ground state of 15F unbound
Goldberg et al. Phys. Rev. C 69 (2004) 031302



6. Phenomenological R matrix Method

Other processes: capture, transfer, inelastic scattering, etc.



Elastic scattering

$E_2, \gamma_2(1)$

$E_1, \gamma_1(1)$

$E_0, \gamma_0(1)$

$$R(E) = \sum_{\lambda=1}^{N} \frac{\gamma_\lambda^2(1)}{E_\lambda - E}$$

Inelastic scattering, transfer

$E_2, \gamma_2(1), \gamma_2(2)$

$E_1, \gamma_1(1), \gamma_1(2)$

$E_0, \gamma_0(1), \gamma_0(2)$

$$R_{ij}(E) = \sum_{\lambda=1}^{N} \frac{\gamma_\lambda(i)\gamma_\lambda(j)}{E_\lambda - E}$$

Pole properties: energy
reduced width in different channels (→ more parameters)
gamma width → capture reactions

6. Phenomenological R matrix Method

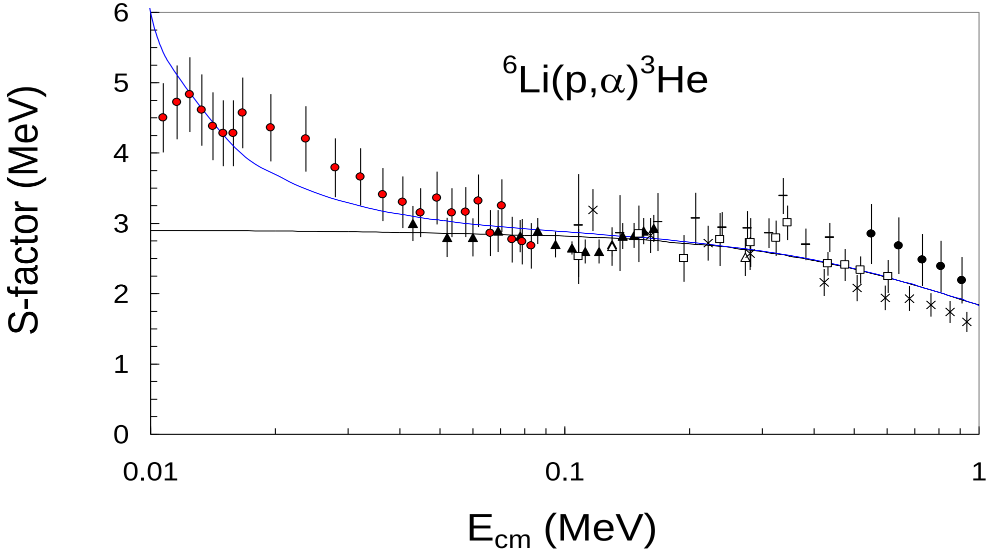
Example of transfer reaction: ^{6}Li(p,α)^{3}He (Nucl. Phys. A639 (1998) 733)

$$R = \begin{pmatrix} R_{pp} & R_{p\alpha} \\ R_{\alpha p} & R_{\alpha\alpha} \end{pmatrix}, \quad \text{with } R_{\alpha p} = R_{p\alpha}$$

Non-resonant reaction: R matrix=constant

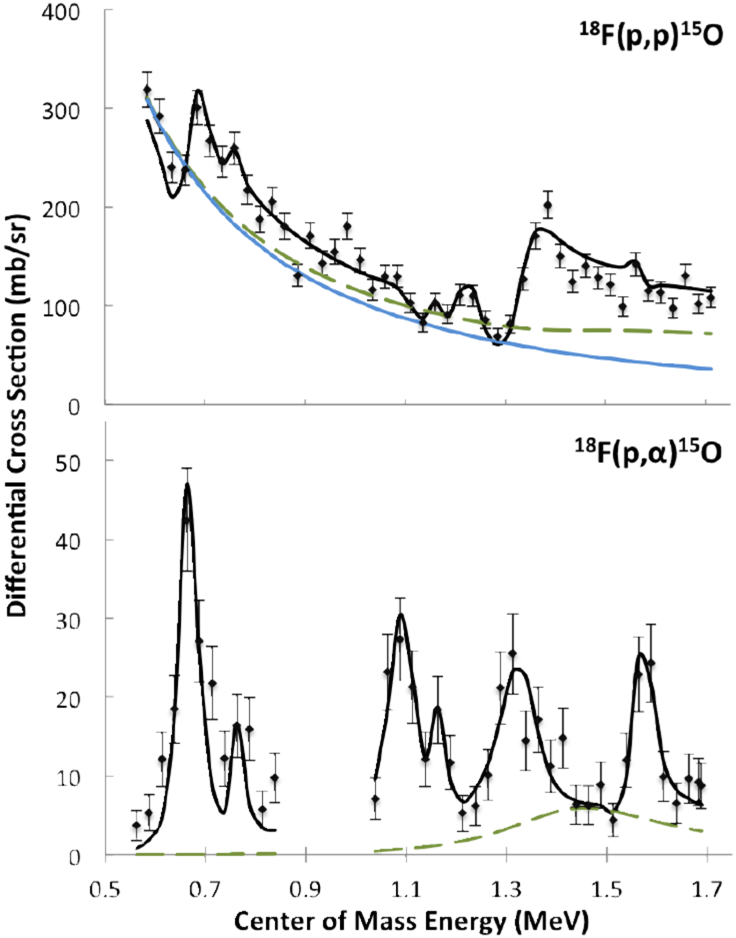
Collision matrix $U = \begin{pmatrix} U_{pp} & U_{p\alpha} \\ U_{\alpha p} & U_{\alpha\alpha} \end{pmatrix}$, deduced from the R matrix

Cross section: $\sigma \sim |U_{p\alpha}|^2$



6. Phenomenological R matrix Method

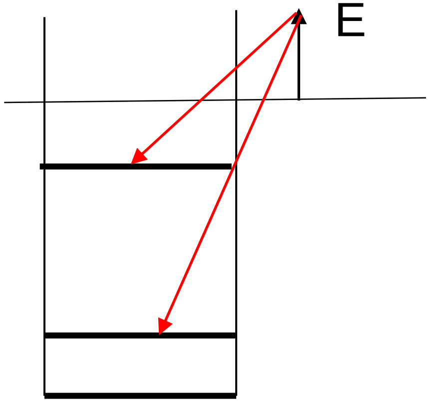
Recent application to $^{18}F(p,p)^{18}F$ and $^{18}F(p,\alpha)^{15}O$
D. Mountford et al, to be published



simultaneous fit of both cross sections
angle: 176°
for each resonance: $J\pi, E_R, \Gamma_p, \Gamma_\alpha$
8 resonances → 24 parameters

6. Phenomenological R matrix Method

Radiative capture



Capture reaction=transition between an initial state at energy E to bound states

Cross section $\sigma_C(E) \sim \left|< \Psi_f \middle| H_\gamma \middle| \Psi_i(E) >\right|^2$

Additional pole parameter: gamma width $\Gamma_{\gamma i}$

$$< \Psi_f | H_\gamma | \Psi_i(E) > = < \Psi_f | H_\gamma | \Psi_i(E) >_{int} + < \Psi_f | H_\gamma | \Psi_i(E) >_{ext}$$

internal part: $< \Psi_f | H_\gamma | \Psi_i(E) >_{int} \sim \sum_{i=1}^{N} \frac{\gamma_i \sqrt{\Gamma_{\gamma i}}}{E_i - E}$

external part:

$$< \Psi_f | H_\gamma | \Psi_i(E) >_{ext} \sim C_f \int_a^\infty W(2k_f r) r^\lambda (I_i(kr) - U O_i(kr)) dr$$

More complicated than elastic scattering!

But: many applications in nuclear astrophysics

7. Conclusions

1. One R-matrix for each partial wave (limited to low energies)
2. Consistent description of resonant and non-resonant contributions (not limited to resonances!)
3. The R-matrix method can be applied in two ways
   a) Calculable R-matrix: to solve the Schrödinger equation
   b) Phenomenological R-matrix: to fit experimental data (low energies, low level densities)

4. Calculable R-matrix
   - Application in many fields of nuclear and atomic physics
   - Efficient to get phase shifts and wave functions of scattering states
   - 3-body systems
   - Stability with respect to the radius is an important test

5. Phenomenological R-matrix
   - Same idea, but the pole properties are used as free parameters
   - Many applications: elastic scattering, transfer, capture, beta decay, etc.