N-Body problem and potentials : toward a Relativistic Dynamics

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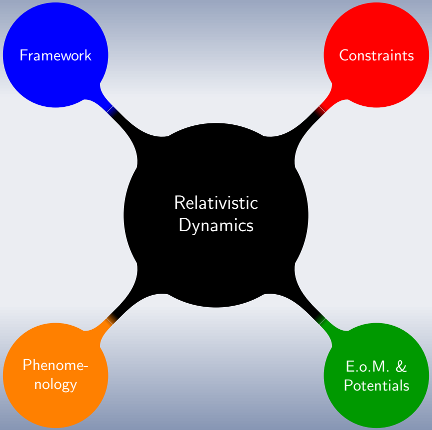
Outline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Constraints
Relativistic Dynamics
Phenome-nology
E.o.M. & Potentials

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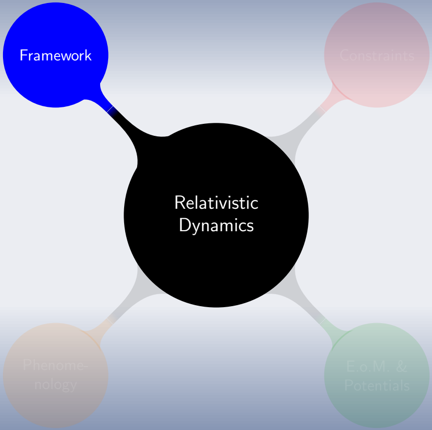
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History

N-body dynamics $\rightarrow$ old problem
for astrophysics, nuclear physics, ...

The classic case is well-studied with a lot of many body potentials, and the ultra-relativistic case is taken without interaction due to the No Interaction Theorem.

(**Currie, Rev. Mod. Phys. 35(1963)**)

No Interaction Theorem

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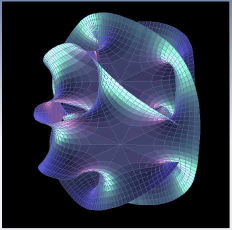
- Invariant world-lines (respect of Poincaré's algebra),
- $8N$ (q^μ, p^μ) independent degrees of freedom,
- Space-time dissociation (clusterisation with potential).

Dimensional reduction I

Particles are in a $8N$ dim. space, but the e.o.m. are just for $6N$. So we have to put $2N$ constraints in our hyperspace (for energy and time).



Equations of motion

$$\frac{\partial \mathbf{q}_i}{\partial t} = \{\mathbf{p}_i, \mathscr{H}\} \qquad \frac{\partial \mathbf{p}_i}{\partial t} = \{\mathbf{q}_i, \mathscr{H}\} \quad \Longrightarrow \quad \frac{\partial q_i^\mu}{\partial \tau} = \{p_i^\mu, \mathcal{H}\} \qquad \frac{\partial p_i^\mu}{\partial \tau} = \{q_i^\mu, \mathcal{H}\}$$

Dimensional reduction II

But attention : in this case $\mathcal{H}$ is not an hamiltonian !

If we use the standard Poisson's Bracket for a system under constraints, we will find wrong e.o.m. In our case, Dirac give a tool for constrained dynamics : the Dirac's Brackets.

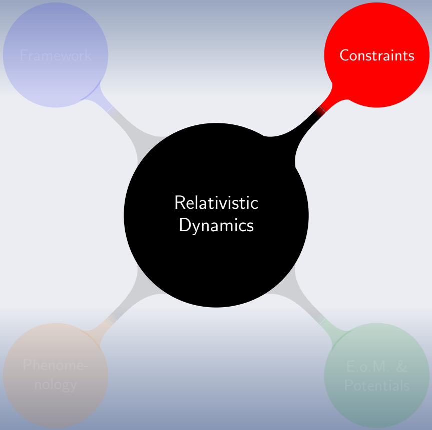
Dirac's Brackets

$$\{A, B\}_D = \{A, B\} - \{A, \phi_i\} C_{ij} \{\phi_j, B\}$$

with a matrix $C_{ij} = \{\phi_i, \phi_j\}^{-1}$ of constraints ϕ.

(**Dirac, Lectures on Quantum Mechanics (1964)**)

Outline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The choice . . .

Now we have to choose these $2N$ constraints.
But the final aim must be to find consistent e.o.m..

My choice of invariant constraints

On-shell mass constraint for energy :

$$K_i = {p_i}^{\mu} {p_i}_{\mu} - {m_i}^2 + V_i = 0$$

Time constraints (interaction or not (χ_i) in a fixed referential (χ_N)) :

$$\chi_i = \sum_{j \neq i} q_{ij}^{\mu} \left(R_{ij} p_{ij\mu} + (1 - R_{ij}) P_{\mu} \right) = 0$$

$$\chi_N = \frac{P^{\mu}}{\sqrt{P^{\mu} P_{\mu}}} \frac{1}{N} \sum_{i=1}^{N} {q_i}_{\mu} - \tau = 0$$

The choice . . .

Now we have to choose these $2N$ constraints.
But the final aim must be to find consistent e.o.m..

My choice of invariant constraints

On-shell mass constraint for energy :

$$\Delta E = 0$$

Time constraints (interaction or not (χ_i) in a fixed referential (χ_N)) :

$$\Delta \tau_{\mathrm{cm}} = 0 \qquad \text{or} \qquad \Delta \tau_{\mathrm{labo}} = 0$$

$$< \tau_i > - \tau = 0$$

. . . which is not a choice

If you remember the Dirac's Bracket,
you know that constraints appear in e.o.m.

For example the kind of energy constraint is easy to guess with the similarity with a "classical" hamiltonian, but we don't want to see the time constraint appear in the e.o.m. !

In the end it is right that we can choose constraints,
but we have to find consistent e.o.m.

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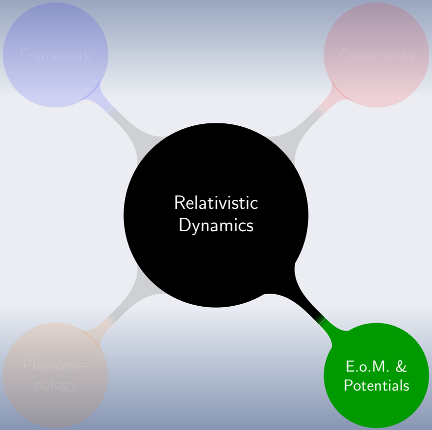
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Classical dynamics

With a standard Hamiltonian such as

$$\mathscr{H} = E = \sqrt{\mathbf{p}^2 + m^2 - V(\mathbf{q})}$$

we get standard e.o.m. :

$$\begin{aligned} \frac{\partial \mathbf{q}_i}{\partial t} &= \{\mathbf{p}_i, \mathscr{H}\} = \frac{\partial \mathscr{H}}{\partial \mathbf{p}_i} = \frac{\mathbf{p}_i}{E_i} \\ \frac{\partial \mathbf{p}_i}{\partial t} &= \{\mathbf{q}_i, \mathscr{H}\} = -\frac{\partial \mathscr{H}}{\partial \mathbf{q}_i} = -\frac{\partial V}{\partial \mathbf{q}_i} + \text{coll.} \end{aligned}$$

Constrained e.o.m.

For a constrained dynamics, especially in the relativistic case, the "hamiltonian" is not the energy. It is the world-line projection operator over the symplectic manyfold ...

Without mathematical words

$$\mathcal{H} \equiv \sum_{i}^{2N} \lambda_i \phi_i = \sum_{i}^{N} \lambda_i K_i = 0$$

Here a consequence of the projection on a referential is the added λ factor. Is is called a Lagrange's Multiplier.

Lagrange's Multiplier I

This λ factor is used to fix the system over our hypersurface.

Global view

$$\lambda_j = C_{2Nj} \quad \text{with} \quad C_{ij} = \{\phi_i, \phi_j\}^{-1}$$

But for N particles we just need N factors.
We can fix for example $\{K_i, K_j\} = 0$.

Constrained view

$$\lambda_j = S_{Nj} \quad \text{with} \quad S_{ij} = \{\chi_i, K_j\}^{-1}$$

Lagrange's Multiplier II

The matrix inversion is a time consumming process for computers. But my time constraint give a matrix which is always invertible.

The other important point is that if we want consistent e.o.m., we need to find a good K_i constraint which respect $\{K_i, K_j\} = 0$. This is called the Komar-Todorov condition.

This condition have consequences on the type of potential which is allowed in Relativistic Dynamics and I will discuss it in the next and last section.

(**Todorov, JINR(1976); Komar, PRD 18(1978)**)

Equations of motion

With the previous constraints we can find the right equations of motion for free particles, and a causal action in case of interaction.

Final e.o.m.

$$\frac{\partial q_i^\mu}{\partial \tau} = 2\ \lambda_i p_i^\mu$$
$$\frac{\partial p_i^\mu}{\partial \tau} = -2 \sum_{k=1}^{N} \lambda_k \ \frac{\partial V_k}{\partial q_{i\,\mu}}$$

Of course we use these e.o.m. only in 6 dim. For free particles $\lambda = 1/2E$, but for interacting particles this is a little bit different.

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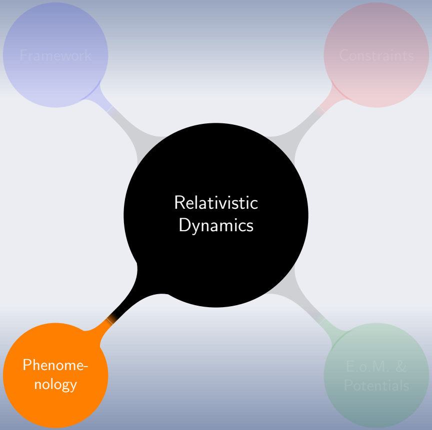
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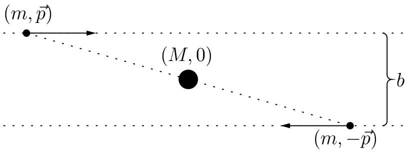
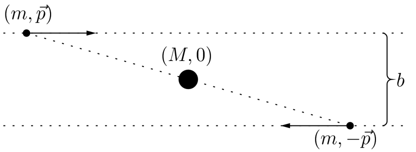
Relativistic differences



Small example with constant masses (one heavy and two light) without interactions but different time constraints

Relativistic differences



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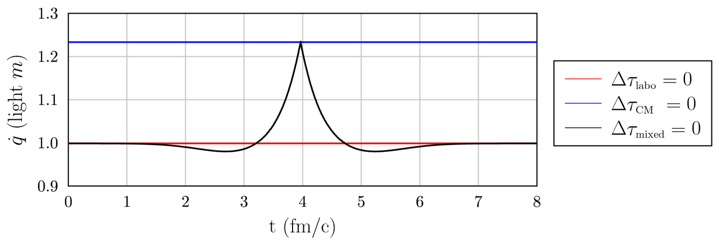
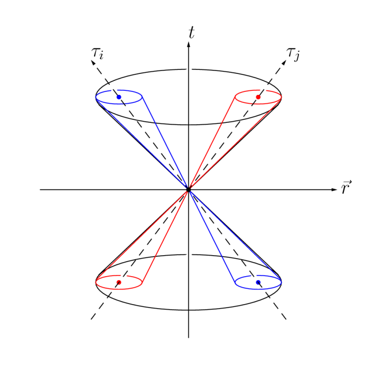


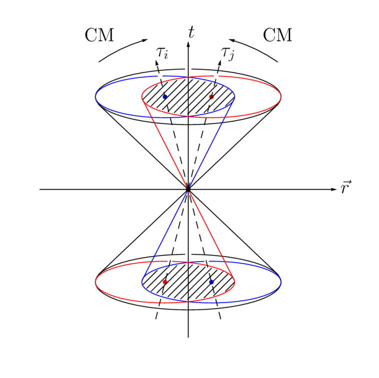
Action viewpoint

We have to minimize action in our system. That leads to maximize time correlation of the forces and collisions.



Action viewpoint

We have to minimize action in our system. That leads to maximize time correlation of the forces and collisions. So the perfect space to do that is the center of mass.



Potential I

In the case with potential we have to be carefull because if we have to respect $\{K_i, K_j\} = 0$ that give :

Komar-Todorov condition

$$p_i^\mu \frac{\partial V_j}{\partial q_i^\mu} - p_j^\mu \frac{\partial V_i}{\partial q_j^\mu} + \{V_i, V_j\} = 0$$

and the potential V_i which respect this equation become difficult to find.

Potential II

The only potential which work with the previous equation is

Solution of potential

$$V_i = \sum_{j \neq i} V_{ij}(q_{Tij}^2)$$

So we use the simple case of 2-body potential.
It can only depend on transverse invariant distance :

Variable for potential

$$q_T{}_{ij}^{\mu} = q_{ij}^{\mu} - \frac{q_{ij}^{\nu} p_{ij\nu}}{p_{ij}^2} p_{ij}^{\mu} \stackrel{\mathrm{CM}}{=} (0, \vec{q}_{ij})$$

Conclusions & Outlooks

- We have a complete consistent method to describe a Relativistic Dynamics,
- We can find the good e.o.m. for free particle in all cases, and for interacting particles with soft potentials and we conserve energy,
- What is happen in case of very strong potential ? Violation of Causality or of the speed of light ?
- Maybe we must investigate research in new constraints ? New potentials ? Question session is opened . . .

Thanks for your attention