# Correlation functions of quantum integrable models : the Bethe ansatz viewpoint

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- "Riemann-Hilbert approach to a generalised sine kernel and applications" *Comm. Math. Phys. 291, 691-761 (2009)*
- "Algebraic Bethe ansatz approach to the asymptotic behavior of correlation functions" *J. Stat. Mech. P04003 (2009)*
- "On the thermodynamic limit of form factors in the massless XXZ Heisenberg chain" *J. Math. Phys. 50, 095209 (2009)*
- "On the thermodynamic limit of particle-holes form factors in the massless XXZ Heisenberg chain" *arXiv:1003.4557*

# The spin-1/2 XXZ Heisenberg chain

The *XXZ* spin-1/2 Heisenberg chain **in a magnetic field** is a quantum interacting model defined on a one-dimensional lattice with $M$ sites, with Hamiltonian,

$$H_{XXZ} = \sum_{m=1}^{M} \{\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta(\sigma_m^z \sigma_{m+1}^z - 1)\} - \mathbf{h} \sum_{m=1}^{M} \sigma_m^z$$

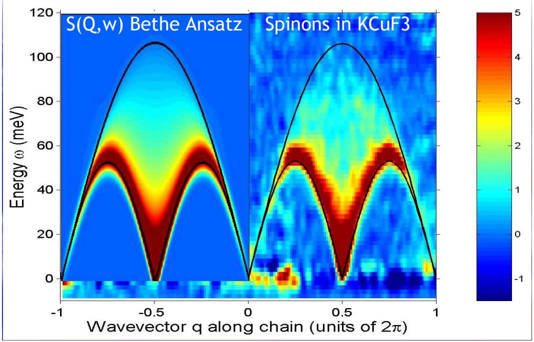
Quantum space of states : $\mathcal{H} = \otimes_{m=1}^{M} \mathcal{H}_m$, $\mathcal{H}_m \sim \mathbb{C}^2$ , $\dim \mathcal{H} = 2^M$.

$\sigma_m^{x,y,z}$ : local spin operators (in the spin-$\frac{1}{2}$ representation) at site $m$
They act as the corresponding Pauli matrices in the space $\mathcal{H}_m$ and as the identity operator elsewhere.

- periodic boundary conditions
- disordered regime, $|\Delta| < 1$ and $h < h_c$

# Why are we interested in correlation functions?

- **Because they define the dynamics of the models**
- **Because they can be measured experimentally**




$S(Q,\omega)$ is the dynamical spin-spin structure factor. The Bethe ansatz curve is computed for a chain of 500 sites (with J.- S. Caux) compared to the experimental curve obtained by A. Tennant in Berlin by neutron scattering. Colors indicate the value of the function $S(Q,\omega)$.

# Correlation function : general strategy

At zero temperature only the ground state $|\omega\rangle$ contributes :

$$g_{12} = \langle\omega|\theta_1\theta_2|\omega\rangle$$

Two main strategies to evaluate such a function:

*(i)* compute the action of local operators on the ground state $\theta_1\theta_2|\omega\rangle = |\tilde{\omega}\rangle$ and then calculate the resulting scalar product:

$$g_{12} = \langle\omega|\tilde{\omega}\rangle$$

*(ii)* insert a sum over a complete set of eigenstates $|\omega_i\rangle$ to obtain a sum over one-point matrix elements (form factor type expansion) :

$$g_{12} = \sum_i \langle\omega|\theta_1|\omega_i\rangle \cdot \langle\omega_i|\theta_2|\omega\rangle$$

# Correlation functions of Heisenberg chain

- **Exact results**
  - Free fermion point $\Delta = 0$: Lieb, Shultz, Mattis, Wu, McCoy, Sato, Jimbo, Miwa ...
  - From 1984: Izergin, Korepin ... (first attempts using Bethe ansatz for general $\Delta$)
  - General $\Delta$: multiple integral representations (for building blocks)
    $\star$ 1992-96 Jimbo, Miwa ... $\rightarrow$ from q-vertex op. and qKZ eq.
    $\star$ 1999 Kitanine, Maillet, Terras $\rightarrow$ from Algebraic Bethe Ansatz
  - Several developments since 2000: Kitanine, Maillet, Slavnov, Terras; Boos, Korepin, Smirnov; Boos, Jimbo, Miwa, Smirnov, Takeyama; Göhmann, Klümper, Seel; Caux, Hagemans, Maillet ...
- **Asymptotic results** $\langle \sigma_1^{\alpha} \sigma_m^{\beta} \rangle \underset{m\to\infty}{\sim}$ ?
  - Luttinger liquid approximation / C.F.T. and finite size effects
    Luther and Peschel, Haldane, Cardy, Affleck, ... Lukyanov, ...

$\hookrightarrow$ **Asymptotic behavior from exact results ?**

# Algebraic Bethe ansatz and correlation functions

1. **Diagonalise the Hamiltonian using ABA**

   → key point : Yang-Baxter algebra $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, $D(\lambda)$

   → $|\,\psi_g\,\rangle = B(\lambda_1)\ldots B(\lambda_N)|\,0\,\rangle$ with $\mathcal{Y}(\lambda_j;\{\lambda\}) = 0$ (Bethe eq.)

2. **Act with local operators on eigenstates**

   → solve the quantum inverse problem (1999):

   $$\sigma_j^{\alpha_j} = f_j^{\alpha_j}(A,B,C,D) = \prod(A,B,C,D)$$

   → use Yang-Baxter commutation relations

3. **Compute the resulting scalar products** (determinant representation)

   → determinant representation for form factors of the finite chain

   → elementary building blocks of correlation functions as multiple integrals in the thermodynamic limit (2000)

4. **Two-point function:** sum up elementary blocks or form factors

   → **Master equation representation** for the finite chain (2005)

5. **Asymptotic analysis of the two-point function (2008-2010):**

   → Series expansion of the Master equation and asymptotic analysis

   → Asymptotic analysis of the form factor series

# The spin-spin correlation functions

$$Q^{\kappa}_{1,m} = \prod_{n=1}^{m} \left( \frac{1+\kappa}{2} + \frac{1-\kappa}{2} \cdot \sigma^z_n \right)$$

Equivalently $Q^{\kappa}_{1,m} = e^{\beta \mathcal{Q}_{1m}}$ with $\mathcal{Q}_{1m} = \frac{1}{2} \sum_{n=1}^{m} (1 - \sigma^z_n)$ and $\kappa = e^{\beta}$.

$$\langle \sigma^z_1 \sigma^z_{m+1} \rangle = 2\, D^2_m\, \partial^2_\kappa \, \langle Q^{\kappa}_{1,m} \rangle \Big|_{\kappa=1} + 2 \langle \sigma^z \rangle - 1 \quad \text{with } D^2_m u_m = u_{m+1} + u_{m-1} - 2u_m$$

- Inverse problem: $Q^{\kappa}_{1,m} = \mathcal{T}_\kappa(0)^m \cdot \mathcal{T}_{\kappa=1}(0)^{-m}$

  with $\mathcal{T}_\kappa(\nu) = A(\nu) + \kappa D(\nu)$ twisted transfer matrix

  $\rightsquigarrow$ $\mathcal{T}_\kappa(\nu) \,|\, \psi_\kappa(\{\mu\}) \,\rangle = \tau_\kappa(\nu|\{\mu\}) \,|\, \psi_\kappa(\{\mu\}) \,\rangle$

  if $\{\mu\}$ is solution of the $\kappa$-twisted Bethe equations $\mathcal{Y}_\kappa(\mu_j|\{\mu\}) = 0$

  $\rightsquigarrow$ $\frac{\mathrm{d}}{\mathrm{d}\mu} \log \mathcal{T}_{\kappa=1}(\mu) \Big|_{\mu=0} \propto H_{XXZ}$

- Act with $\mathcal{T}_\kappa(0)^m \cdot \mathcal{T}_{\kappa=1}(0)^{-m}$ on $|\, \psi_g \,\rangle$ or sum over $\kappa$-deformed form-factors

  $\Longrightarrow$ Master equation

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# Master equation for the finite chain

$$\langle Q_{1,m}^{\kappa}\rangle = \frac{\langle\, \psi_g \,|\mathcal{T}_{\kappa}(0)^m \cdot \mathcal{T}_{\kappa=1}(0)^{-m}|\, \psi_g\,\rangle}{\langle \psi_g|\psi_g\rangle}$$

$\longrightarrow$ $\langle Q_{1,m}^{\kappa}\rangle$ polynomial in $\kappa$

$\longrightarrow$ for $\kappa$ small enough the spectrum of $\mathcal{T}_{\kappa}$ is simple, well separated from the one at $\kappa = 1$, described by $\kappa$-twisted Bethe equations $\mathcal{Y}_{\kappa}(\mu_j|\{\mu\}) = 0$ , and $\kappa$-twisted Bethe states $|\,\psi_{\kappa}(\{\mu\})\,\rangle$ form a complete basis

$$\langle Q_{1,m}^{\kappa}\rangle = \sum_{\substack{\{\mu\} \text{ solutions of} \\ \text{twisted Bethe eq.}}} \frac{\langle\psi_g|\psi_{\kappa}(\{\mu\})\rangle \cdot \langle\psi_{\kappa}(\{\mu\})|\psi_g\rangle}{\langle\psi_{\kappa}(\{\mu\})|\psi_{\kappa}(\{\mu\})\rangle \cdot \langle\psi_g|\psi_g\rangle} \cdot \frac{\tau_{\kappa}(0|\{\mu\})^m}{\tau(0|\{\lambda\})^m}$$

$\longrightarrow$ can be rewritten as a multiple contour integral around the solutions $\{\mu\}$ of the $\kappa$-twisted Bethe equations, with the product of these twisted Bethe equations $\mathcal{Y}_{\kappa}(z_j|\{z\})$ in denominator (the norm squared of these states is related to the Jacobian of the $\kappa$-twisted Bethe equations).

$$\langle Q_{1,m}^{\kappa}\rangle = \frac{1}{N!} \oint_{\substack{\Gamma(\{\mu\}) \\ \text{solutions of} \\ \text{twisted Bethe eq.}}} \frac{\mathrm{d}^N z}{(2\pi i)^N} \prod_{j=1}^{N} \left[ e^{im[p_0(z_j)-p_0(\lambda_j)]} \frac{d(z_j)}{d(\lambda_j)} \right] \frac{\left[\det_N \Omega_{\kappa}(\{z\},\{\lambda\}|\{z\})\right]^2}{\prod_{j=1}^{N} \mathcal{Y}_{\kappa}(z_j|\{z\}) \cdot \det_N \frac{\partial \mathcal{Y}(\lambda_j|\{\lambda\})}{\partial \lambda_k}}$$

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**$\rightsquigarrow$ Different forms for master equation (two ways of writing scalar products):**

• with simple poles at $\{\lambda\}$ (parameters for the ground state) and $\{\xi\}$ (inhomogeneity parameters) + poles at $\{\mu\}$ solutions of $\kappa$-twisted Bethe equations (initial form obtained from multiple integrals)

• with **double poles** at $\{\lambda\}$ (parameters for the ground state) + poles at $\{\mu\}$ solutions of $\kappa$-twisted Bethe equations (form we use here)

$$\langle Q^{\kappa}_{1,m}\rangle = \frac{(-1)^N}{N!}\oint_{\Gamma(\{\lambda\})}\frac{\mathrm{d}^N z}{(2\pi i)^N}\prod_{j=1}^{N}\left[e^{im[p_0(z_j)-p_0(\lambda_j)]}\frac{d(z_j)}{d(\lambda_j)}\right]\frac{\left[\det_N \Omega_{\kappa}(\{z\},\{\lambda\}|\{z\})\right]^2}{\prod_{j=1}^{N}\mathcal{Y}_{\kappa}(z_j|\{z\})\det_N\frac{\partial\mathcal{Y}(\lambda_j|\{\lambda\})}{\partial\lambda_k}}$$

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# The series expansion: thermodynamic limit

⇝ **Single out the poles** "$z_j = \lambda_k$":

$$\det_N \Omega_\kappa(\{z\},\{\lambda\}|\{z\}) = \det_N \left[\frac{1}{\sinh(z_j-\lambda_k)}\right] \cdot \det_N T_\kappa(\{z\},\{\lambda\}|\{z\})$$

⇝ Reorganize and expand determinants → **Series expansion**

⇝ **Thermodynamic limit** $(N, M \to \infty,\ N/M \to D,\ \{\lambda\} \to \rho(\lambda)$ on $[-q,q])$:

$$\langle e^{\beta \mathcal{Q}_{1m}} \rangle = \sum_{n=0}^{+\infty} \frac{1}{n!} \int_{-q}^{q} \frac{d^n\lambda}{(2i\pi)^n} \oint_{\Gamma([-q,q])} \frac{d^n z}{(2i\pi)^n} \prod_{\ell=1}^{n} \frac{e^{im[p_0(z_\ell)-p_0(\lambda_\ell)]}}{\sinh(z_\ell-\lambda_\ell)} \times \det_n \left[\frac{1}{\sinh(z_k-\lambda_j)}\right] \cdot \mathcal{F}_n^{(\kappa)} \begin{pmatrix} \{\lambda\} \\ \{z\} \end{pmatrix}$$

with $\mathcal{F}_n^{(\kappa)}$ symmetric in $\{\lambda\}$ and $\{z\}$ + satisfy reduction properties at "$z_j = \lambda_k$"

⋆ if $\mathcal{F}_n^{(\kappa)} = \prod_{i=1}^{n} [\varphi(\lambda_i)\, e^{g(z_i)}]$ decoupled → **Fredholm determinant** :

$$\langle e^{\beta \mathcal{Q}_{1m}} \rangle = \sum_{n=0}^{+\infty} \frac{1}{n!} \int_{-q}^{q} \frac{d^n\lambda}{(2i\pi)^n} \det_n[V(\lambda_j,\lambda_k)]$$

$$V(\lambda,\mu) = \varphi(\lambda) e^{g(\lambda)} \frac{\sin\left\{\frac{m}{2}[p_0(\lambda)-p_0(\mu)] - \frac{i}{2}[g(\lambda)-g(\mu)]\right\}}{\pi \sinh(\lambda-\mu)}$$

⋆ $\mathcal{F}_n^{(\kappa)}$ not decoupled ?

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# The series expansion: decomposition into cycle integrals

**→ Using standard cycle expansion of the determinant**

Cycles of element of the symmetry group are labelled by $(s, p)$, cycle variables labelled by $(s, p, j)$, $1 \leq p \leq \ell_s$, $1 \leq j \leq s$, $s$=length of a cycle, $\ell_s$=number of cycles of length $s$

$$\langle e^{\beta \mathcal{Q}_m}\rangle = \sum_{n=0}^{+\infty} \sum_{\substack{\ell_1,\dots,\ell_n=0\\ \Sigma k\ell_k=n}} C(n|\{\ell\}) \left\{ \prod_{s=1}^{n}\prod_{p=1}^{\ell_s} \mathfrak{I}_{(s,p)} \right\} [\mathcal{F}_n^{(\kappa)}]$$

Each $\mathfrak{I}_{s,p}$ integrates over the variables $\lambda_{s,p,j}$ and $z_{s,p,j}$ with $1 \leq j \leq s$.

$$\mathfrak{I}_s[\mathcal{G}_s] = \oint_{\Gamma([-q,q])} \frac{\mathrm{d}^s z}{(2i\pi)^s} \int_{-q}^{q} \frac{\mathrm{d}^s\lambda}{(2i\pi)^s}\, \mathcal{G}_s\begin{pmatrix}\{\lambda\}\\ \{z\}\end{pmatrix} \prod_{j=1}^{s} \frac{\exp\{im[p_0(\lambda_j)-p_0(z_j)]\}}{\sinh(z_j-\lambda_j)\sinh(z_j-\lambda_{j+1})}$$

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# Cycle integrals and generalized sine kernel

$$\mathfrak{I}_s[\mathcal{G}_s] = \oint\limits_{\Gamma([-q,q])} \frac{\mathrm{d}^s z}{(2i\pi)^s} \int\limits_{-q}^{q} \frac{\mathrm{d}^s \lambda}{(2i\pi)^n} \, \mathcal{G}_s\begin{pmatrix}\{\lambda\}\\ \{z\}\end{pmatrix} \prod_{j=1}^{s} \frac{\exp\left\{im\left[p_0(\lambda_j) - p_0(z_j)\right]\right\}}{\sinh(z_j - \lambda_j)\sinh(z_j - \lambda_{j+1})}$$

with $\mathcal{G}_s$ symmetric separately in $\{\lambda\}$ and in $\{z\}$

- if $\mathcal{G}_s\begin{pmatrix}\{\lambda\}\\ \{z\}\end{pmatrix} = \prod_{i=1}^{s}\left[\varphi(\lambda_i)\, e^{g(z_i)}\right]$ then $\mathfrak{I}_s[\mathcal{G}_s]$ can be obtained in terms of the **Fredholm determinant** of a **generalized sine kernel** :

$$\mathfrak{I}^{(s)}[\mathcal{G}_s] = \int\limits_{-q}^{q} \mathrm{d}^n\lambda \prod_{j=1}^{s} V^{(\varphi,g)}(\lambda_j,\lambda_{j+1}) = \frac{(-1)^{s-1}}{(s-1)!}\frac{\partial^s}{\partial\gamma^s}\log\det[I + \gamma V^{(\varphi,g)}]\Big|_{\gamma=0}$$

$$V^{(\varphi,g)}(\lambda,\mu) = F(\lambda)\frac{\sin\left\{\frac{m}{2}\left[p_0(\lambda) - p_0(\mu)\right] - \frac{i}{2}\left[g(\lambda) - g(\mu)\right]\right\}}{\pi\sinh(\lambda-\mu)}$$

with $F(\lambda) = \varphi(\lambda)e^{g(\lambda)}$

- **density theorem** in the general case: $\mathcal{G}_s\begin{pmatrix}\{\lambda\}\\ \{z\}\end{pmatrix} = \sum_{\ell=1}^{\infty}\prod_{i=1}^{s}\left[\varphi_\ell(\lambda_i)\, e^{g_\ell(z_i)}\right]$

$\hookrightarrow$ Analyze the asymptotic behavior of the generalized sine kernel and apply the density procedure to get the asymptotic of $\mathfrak{I}_s$

# Asymptotic behavior of cycle integrals

- **Asymptotics of the generalized sine-kernel**
  $\to$ **Matrix Riemann-Hilbert Problems** (generalization of the procedure of Deift, Its, Zhou (1997) for the sine-kernel)
- **Application to cycle integrals**
  $\to$ take the $n^{th}$ $\gamma$-derivative
  $\to$ specialize to $V^{(\varphi,g)}$
  $\to$ apply the density procedure (corrections remain corrections)

$$\mathfrak{I}_s[\mathcal{G}_s] = H_s[\mathcal{G}_s] + D_s[\mathcal{G}_s] + O_s[\mathcal{G}_s]$$

$$H_s[\mathcal{G}_s] = \frac{1}{2\pi i}\int_{-q}^{q} \mathrm{d}\lambda \Big\{ imp_0'(\lambda) - b_s \log\big(m \sinh(2q) p_0'(\lambda)\big) \times[\delta(\lambda+q)+\delta(\lambda-q)]\Big\} \mathcal{G}_s\begin{pmatrix}\lambda,\dots,\lambda\\ \lambda,\dots,\lambda\end{pmatrix} + C[\mathcal{G}_s]$$

$$D_s[\mathcal{G}_s] = \int_{-q}^{q} \frac{\mathrm{d}\lambda}{2i\pi}\,\partial_\epsilon\, \mathcal{G}_s\begin{pmatrix}\lambda,\lambda,\dots,\lambda\\ \lambda+\epsilon,\lambda,\dots,\lambda\end{pmatrix}\Bigg|_{\epsilon=0} + \cdots \text{ (derivative)}$$

$$O_s[\mathcal{G}_s] = \text{terms of order o}(1) \text{ (contains oscillating contributions } e^{irmp_0(\pm q)})$$

# Asymptotic summation of the series

$$\langle e^{\beta \mathcal{Q}_{1m}} \rangle = \sum_{n=0}^{+\infty} \sum_{\substack{\ell_1,\dots,\ell_n=0 \\ \sum k\ell_k = n}} C(n|\{\ell\}) \prod_{s=1}^{n} \prod_{p=1}^{\ell_s} \left\{ H_{s,p} + D_{s,p} + O_{s,p} \right\} \left[ \mathcal{F}_n^{(\kappa)} \right]$$

Sum up successively (use binomial formula) $H_s$, then $D_s$, then $O_s$
+ use the **reduction properties of $\mathcal{F}_n^{(\kappa)}$ at $z_j = \lambda_k$**

$\longrightarrow$ the series of $H_s$ exponentiates
$\longrightarrow$ the series of successive actions of $D_s$ is a continuous generalization of the multiple Lagrange series : its sum is expressed in terms of a solution of an integral equation
$\longrightarrow$ sum-up $O_s$ pertubatively

# Asymptotic summation of the series

The series we have to sum up is in fact a functional version of the standard Lagrange series of the type

$$G_0^{(h)} = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{d}{d\epsilon} + h \right)^n \left( F(\epsilon)\phi^n(\epsilon) \right) \Big|_{\epsilon=0}$$

where $F(z)$ and $\phi(z)$ are some functions holomorphic in a vicinity of the origin. If the series is convergent, then it can be summed up in terms of the solution of the equation $z - \phi(z) = 0$ and the sum is given by

$$G_0^{(h)} = \frac{F(z)\, e^{hz}}{1 - \phi'(z)}$$

In the correlation function case, $z$ becomes a function and $\Phi(z)$ an integral operator acting on this function; hence the summation is given as the value of some functional in a point determined by an integral equation.

# Results

## Generating function

$$\langle e^{\beta \mathcal{Q}_{1m}} \rangle = \underbrace{G^{(0)}(\beta, m)[1 + o(1)]}_{\text{non-oscillating terms}} + \underbrace{\sum_{\sigma=\pm} G^{(0)}(\beta + 2i\pi\sigma, m)[1 + o(1)]}_{\text{oscillating terms}}$$

$$G^{(0)}(\beta, m) = C(\beta)\, e^{m\beta D}\, m^{\frac{\beta^2}{2\pi^2} Z(q)^2}$$

- $Z(\lambda)$ is the dressed charge $\quad Z(\lambda) + \int_{-q}^{q} \frac{\mathrm{d}\mu}{2\pi} K(\lambda - \mu)\, Z(\mu) = 1$
- $D$ is the average density $\quad D = \int_{-q}^{q} \rho(\mu)\mathrm{d}\mu = \frac{1 - \langle \sigma^z \rangle}{2} = \frac{p_F}{\pi}$
- The coefficient $C(\beta)$ is given as the ratio of four Fredholm determinants.
- sub-leading oscillating terms restore the $2\pi i$-periodicity in $\beta$

## 2-point function

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle = (2D - 1)^2 - \frac{2Z(q)^2}{\pi^2 m^2} + 2|F_{\sigma^z}|^2 \cdot \frac{\cos(2mp_F)}{m^{2Z(q)^2}} + o\left(\frac{1}{m^2}, \frac{1}{m^{2Z(q)^2}}\right)$$

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# Form factors strike back

## The umklapp form factor

$$\lim_{N,M\to\infty}\left(\frac{M}{2\pi}\right)^{2\mathcal{Z}^2}\frac{|\langle\psi(\{\mu\})|\sigma^z|\psi(\{\lambda\})\rangle|^2}{\|\psi(\{\mu\})\|^2\cdot\|\psi(\{\lambda\})\|^2}=|F_{\sigma^z}|^2.$$

with

$$2\mathcal{Z}^2=Z(q)^2+Z(-q)^2$$

- $\{\lambda\}$ are the Bethe parameters of the ground state
- $\{\mu\}$ are the Bethe parameters for the excited state with one particle and one hole on opposite sides of the Fermi boundary (umklapp type excitation).

$\hookrightarrow$ Higher terms in the asymptotic expansion will involve $n$ - particle/holes form factors corresponding $2np_F$ oscillations
$\hookrightarrow$ Properly normalized form factors should be related to the corresponding amplitudes

$\hookrightarrow$ Analyze the asymptotic behavior of the correlation function directly from the form factor series!

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$\hookrightarrow$ Analyze the asymptotic behavior of the correlation function directly from the form factor series!

# The form factor series

↪ Compute normalized form factors for large size chain

$$\mathcal{F}^{(s)}_{\psi\,\psi'}(m) = \frac{\langle\, \psi \,|\, \sigma^s_m \,|\, \psi' \,\rangle}{\|\psi\| \cdot \|\psi'\|}\;, \qquad s = x, y, z,$$

↪ Analyze the form factor series for large distance for $M \to \infty$

$$\frac{\langle \psi_g \,|\, \sigma^s_m\, \sigma^{s'}_{m'} \,|\, \psi_g \rangle}{\langle \psi_g \,|\, \psi_g \rangle} = \sum_{|\,\psi'\rangle} \mathcal{F}^{(s)}_{\psi_g\,\psi'}(m)\, \mathcal{F}^{(s')}_{\psi'\,\psi_g}(m').$$

↪ We need to control states that will contribute to the leading behavior of the series in the limit $M \to \infty$ and also to compute the corresponding form factors and their behavior in this limit

↪ The form factor series is analogous to a multiple highly oscillatory integral due to the exponent of distance. Hence, the leading asymptotic behavior comes from the ends of the summation interval (the Fermi surface) plus an eventual saddle point of the exponent that will appear in the time dependent case

# The ground state

## Logarithmic Bethe equations

$$Mp_0(\lambda_j) - \sum_{k=1}^{N} \vartheta(\lambda_j - \lambda_k) = 2\pi n_j, \qquad j = 1, \ldots, N.$$

$p_0(\lambda)$ and $\vartheta(\lambda)$ the bare momentum and phase,

$$p_0(\lambda) = i \log\left(\frac{\sinh(\frac{i\zeta}{2} + \lambda)}{\sinh(\frac{i\zeta}{2} - \lambda)}\right) \qquad \vartheta(\lambda) = i \log\left(\frac{\sinh(i\zeta + \lambda)}{\sinh(i\zeta - \lambda)}\right)$$

$0 < \zeta < \pi$, $\cos\zeta = \Delta$ and $n_j$, $-M/2 < n_j \leq M/2$, are integers (for $N$ odd) or half-integers (for $N$ even).

## Ground state Bethe roots

$\lambda_j$, $j = 1, \ldots, N$, with $n_j = j - (N+1)/2$, $N$ the number of down spins fixed by the overall magnetic field $h$. Thermodynamic limit $N, M \to \infty$, $N/M$ tends to some fixed density $D$ and $\lambda_j \in [-q, q]$ with density $\rho(\omega)$

# The counting function

## Ground state counting function

$$\widehat{\xi}(\omega) = \frac{1}{2\pi} p_0(\omega) - \frac{1}{2\pi M} \sum_{k=1}^{N} \vartheta(\omega - \lambda_k) + \frac{N+1}{2M} \quad , \ \widehat{\xi}(\lambda_j) = j/M \quad , \ j = 1, \dots, N$$

$$\rho(\lambda) + \frac{1}{2\pi} \int_{-q}^{q} K(\lambda - \mu)\, \rho(\mu)\, d\mu = \frac{1}{2\pi} p_0'(\lambda), \quad \text{with} \quad K(\lambda) = \vartheta'(\lambda).$$

$$p(\lambda) = 2\pi \int_{0}^{\lambda} \rho(\mu)\, d\mu, \qquad \xi(\lambda) = [p(\lambda) + p(q)]/2\pi, \qquad \pi D = p(q).$$

## Dressed phase $\phi(\lambda, \nu)$

$$\phi(\lambda,\nu) + \frac{1}{2\pi} \int_{-q}^{q} K(\lambda - \mu)\, \phi(\mu,\nu)\, d\mu = \frac{1}{2\pi} \vartheta(\lambda - \nu)$$

# The excited states

**The counting function of twisted excited states**

$$\widehat{\xi}_\kappa(\omega) = \frac{1}{2\pi} p_0(\omega) - \frac{1}{2\pi M} \sum_{k=1}^{N_\kappa} \vartheta(\omega - \mu_{\ell_k}) + \frac{N_\kappa + 1}{2M} - \frac{\alpha}{M}$$

Excited state : obtained by removing the solutions $\widehat{\xi}_\kappa\,(\mu_{h_a}) = h_a/M$ and replacing them by the solutions $\widehat{\xi}_\kappa\,(\mu_{p_a}) = p_a/M$. Namely $\mu_{h_a}$ stands for the rapidities of the holes and $\mu_{p_a}$ stands for those of the particles and $\mu_j$ solution to $\widehat{\xi}_\kappa\,(\mu_j) = j/M$
$\hookrightarrow$ Characterized by the shift function $\widehat{F}(\omega) = \widehat{F}(\omega|\{\mu_p\}|\{\mu_h\})$

$$\widehat{F}(\omega) = M\big(\widehat{\xi}(\omega) - \widehat{\xi}_\kappa(\omega)\big).$$

The shift function describes the spacing between the root $\lambda_j$ for the ground state in the $N$ sector and the parameters $\mu_j$ defined by $\widehat{\xi}_\kappa\,(\mu_j) = j/M$:

$$\mu_j - \lambda_j = \frac{F(\lambda_j)}{\rho(\lambda_j)M} + O(M^{-2}),$$

# The thermodynamic shift function

Recall that we consider the excited states in the $N_\kappa$ sector with $N_\kappa = N$ (for form factors of $\sigma^z$) and $N_\kappa = N+1$ (for form factors of $\sigma^+$). Respectively we should distinguish between two shift functions $F^{(z)}(\lambda)$ and $F^{(+)}(\lambda)$ corresponding to these two cases. Generically the shift function $F(\lambda)$ satisfies the integral equation

$$F(\lambda)+\int_{-q}^{q} K(\lambda-\mu)F(\mu)\,\frac{d\mu}{2\pi} = \alpha+\frac{\delta N}{2}\left[1-\frac{\vartheta(\lambda-q)}{\pi}\right]+\frac{1}{2\pi}\sum_{k=1}^{n}\left[\vartheta(\lambda-\mu_{p_k})-\vartheta(\lambda-\mu_{h_k})\right]$$

where $\delta N = N - N_\kappa$. We have :

$$F^{(z)}(\lambda) = \alpha Z(\lambda) + \sum_{k=1}^{n}\phi(\lambda,\mu_{p_k}) - \sum_{k=1}^{n}\phi(\lambda,\mu_{h_k})\;.$$

$$F^{(+)}(\lambda) = \left(\alpha-\frac{1}{2}\right) Z(\lambda) + \sum_{k=1}^{n}\phi(\lambda,\mu_{p_k}) - \sum_{k=1}^{n}\phi(\lambda,\mu_{h_k}) + \phi(\lambda,q)\;.$$

# The main result

## Asymptotic behavior of form factors

$$\mathcal{F}^{(z)}_{\psi_g\,\psi'}(m')\cdot\mathcal{F}^{(z)}_{\psi'\,\psi_g}(m)\sim\delta_{N,N_\kappa}\;M^{-\theta_{zz}}\;e^{i\mathcal{P}_{ex}(m-m')}\;\partial^2_\alpha\,\mathcal{S}_{zz}\,\mathcal{D}_{zz}\Big|_{\alpha=0}$$

$$\mathcal{P}_{ex}=2\pi\alpha D+\sum_{j=1}^{n}\left[p(\mu_{p_j})-p(\mu_{h_j})\right]$$

The smooth part $\mathcal{S}_{zz}$ : depends continuously on the rapidities $\mu_{p_j}$ and $\mu_{h_j}$ of the particles and holes. The discrete part $\mathcal{D}_{zz}$ also depends on the set of integers appearing in the logarithmic Bethe Ansatz equations for the excited state and The exponents $\theta_{zz}$ computed explicitly in terms of shift function

$\hookrightarrow$ If particles or holes approach the Fermi surface, the discrete structure of the form factors can no longer be neglected: a microscopic (of order $1/M$) deviation of a particle (or hole) rapidity leads to a macroscopic change in $\mathcal{D}_{zz}$.

In the thermodynamic limit, we say that a given excited state belongs to the $\mathcal{E}_r$ class if it contains $n_p^{\pm}$ particles, resp. $n_h^{\pm}$ holes, with rapidities equal to $\pm q$ such that

$$n_p^+-n_h^+=n_h^--n_p^-=r,\qquad r\in\mathbb{Z}.$$

# The amplitudes

$$\mathcal{D}_{zz} = \mathcal{D}^{(z)}\left[F_r^{(z)}\right] \frac{G^2\left(1+F_+^{(z)}\right) G^2\left(1-F_-^{(z)}\right)}{G^2\left(1+F_{r,+}^{(z)}\right) G^2\left(1-F_{r,-}^{(z)}\right)} \left(\frac{\sin(\pi F_{r,+}^{(z)})}{\pi}\right)^{2n_h^+} \left(\frac{\sin(\pi F_{r,-}^{(z)})}{\pi}\right)^{2n_h^-}$$
$$\times R_{n_p^+,n_h^+}(\{p^+\},\{h^+\}|F_+^{(z)})\, R_{n_p^-,n_h^-}(\{p^-\},\{h^-\}|-F_-^{(z)})$$

With $F_r^{(z/+)}(\lambda) = F^{(z/+)}(\lambda) + r$, $F_{r,\pm}^{(z/+)} = F_r^{(z/+)}(\pm q)$ and $G$ the Barnes function.

$$\Gamma\left(\begin{array}{ccc} a_1 , & \ldots , & a_\ell \\ b_1 , & \ldots , & b_j \end{array}\right) = \prod_{k=1}^{\ell} \Gamma(a_k) \cdot \prod_{k=1}^{j} \Gamma(b_k)^{-1}$$

$$R_{n,m}(\{p\},\{h\}|F) = \frac{\prod_{j>k}^{n} (p_j - p_k)^2 \prod_{j>k}^{m} (h_j - h_k)^2}{\prod_{j=1}^{n}\prod_{k=1}^{m} (p_j + h_k - 1)^2}\, \Gamma^2\left(\begin{array}{cc} \{p_k + F\} , & \{h_k - F\} \\ \{p_k\} , & \{h_k\} \end{array}\right)$$

$$\mathcal{S}_{zz} = \frac{2}{\pi^2}\sin^2\left(\frac{\mathcal{P}_{ex}}{2}\right) \cdot \mathcal{A}_n^{(z)} \cdot e^{C_n^{(z)}},$$

# Critical exponents

$$\theta_{zz}(r) = \left(F^{(z)}_{r,+}\right)^2 + \left(F^{(z)}_{r,-}\right)^2$$

$$F^{(z)}_r(\lambda) = (\alpha + r) Z(\lambda)$$

Therefore, for $\alpha = 0$ :

$$\theta_{zz}(r) = 2r^2 Z^2(q), \quad |r| = 1, 2, \dots$$

These numbers coincide with the critical exponents in $\langle \sigma^z_1 \sigma^z_{m+1} \rangle$. Similarly :

$$\theta_{+-}(r) = Z^{-2}(q)/2 + 2r^2 Z^2(q), \quad |r| = 1, 2, \dots$$

Again, these numbers coincide with the critical exponents in $\langle \sigma^+_1 \sigma^-_{m+1} \rangle$.

$\hookrightarrow$ Asymptotic summation of the form factor series will just produce the transmutation from the size $M$ to the distance $m$ as $M \to 2\pi m$ which explain the equality of the exponents; in fact this effect is mainly "free fermion" like as it exists (although simplified) in that case too and can be somehow "easily" deformed to the interacting situation. This deformation involves again Fredholm determinant of the GSK.

## Further results and open questions

- Summation of the form factor series for various correlation functions
- Time dependent case for the Bose gas
- Time dependent case for XXZ : needs careful treatment of bound-states
- Asymptotics for large distances in the temperature case (contact with QTM method)
- Other models like Sine-Gordon?