Quantum Sine(h)-Gordon Model

and

Classical Integrable Equations

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math-ph/0013167 (with A. Zamolodchikov)

In over three decades of study of quantum integrable systems, a remarkable (and largely mysterious) relation to classical integrable equations was observed in a number of different context

- Spin-spin correlation function in the Ising Field Theory:

  Barouch, Tracy, McCoy (1975)

- Self-avoiding polymer problem:

  Cecotti, Fendley, Intriligator, Vafa (1992)

  Fendley, Saleur (1992)

  Al.Zamolodchikov (1994)

Special Painlevé III

$$\frac{\mathrm{d}^2 \mathrm{f}}{\mathrm{d}\tau^2} + \frac{1}{\tau}\frac{\mathrm{d}\mathrm{f}}{\mathrm{d}\tau} - \frac{1}{2}\ \sinh(2f) = 0$$

Modified Sinh-Gordon (MShG)

$$\partial_z \partial_{\bar{z}} \eta + \sigma \left[\mathrm{e}^{2\eta} - p(z)\, p(\bar{z})\ \mathrm{e}^{-2\eta} \right] = 0 \qquad\qquad (\sigma = \pm 1)$$

- Constant mean curvature surfaces in $\mathbb{R}_3$, $\mathbb{S}_3$ $(\sigma > 0)$ and in $\mathbb{A}dS_3$ $(\sigma < 0)$

- Minimal surfaces in $\mathbb{A}dS_3$ $(\sigma < 0)$

  Alday, Maldacena (2007, 2009): Strong coupling scattering amplitudes in the $4D$, $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. For $2n$-gluons:

  Worldsheet: whole complex plane z and

$$p(z) = z^{n-2} + m_{n-4}\, z^{n-4} + \ldots m_0$$
$$\bar{p}(\bar{z}) = \bar{z}^{n-2} + m^*_{n-4}\, \bar{z}^{n-4} + \ldots m^*_0$$

- BPS states of $4D$ $\mathcal{N} = 2$ theories and wall crossing Gaiotto, Moore, Neitzke (2008, 2009)

MShG = ShG (locally)

$$w = \int \mathrm{d}z \ \sqrt{p(z)} \ , \qquad \widehat{\eta}(w, \bar{w}) = \eta - \frac{1}{4} \ \log(p\bar{p})$$

$$\partial_z \partial_{\bar{z}} \eta - \mathrm{e}^{2\eta} + p(z)\, \bar{p}(\bar{z}) \ \mathrm{e}^{-2\eta} = 0 \qquad \longrightarrow \qquad \partial_w \partial_{\bar{w}} \widehat{\eta} - \mathrm{e}^{2\widehat{\eta}} + \mathrm{e}^{-2\widehat{\eta}} = 0$$

The problem

$$\partial_z \partial_{\bar z} \eta - \mathrm{e}^{2\eta} + p(z)\, p(\bar z)\ \mathrm{e}^{-2\eta} = 0$$

$$p(z) = z^{2\alpha} - s^{2\alpha} \qquad (\, \alpha, s > 0\,)$$

Family of solutions parameterized by real $l \in \left[-\frac{1}{2}, \frac{1}{2}\right]$**:**

- η **is a single-valued solutions of MShG on the cone** $\mathbb{C}_{\frac{\pi}{\alpha}}$ **with an apex angle** $\frac{\pi}{\alpha}$**:**

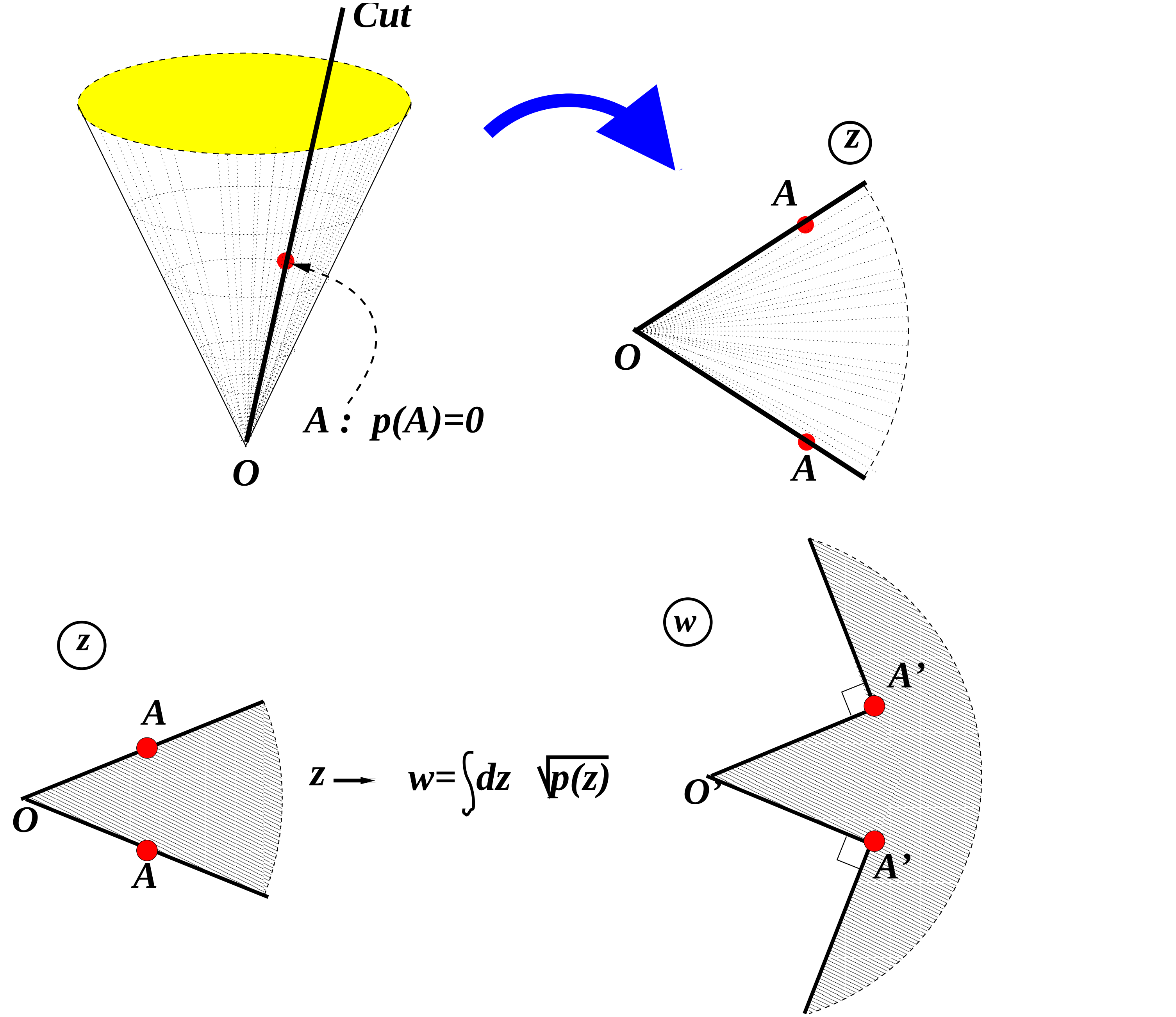
$$\mathbb{C}_{\frac{\pi}{\alpha}}\ : \qquad (z,\, \bar z) \sim \left(\mathrm{e}^{\frac{\mathrm{i}\pi}{\alpha}}\ z,\, \mathrm{e}^{-\frac{\mathrm{i}\pi}{\alpha}}\ \bar z\right), \qquad 0 \le |z| < \infty\, .$$

- **Large-**$|z|$ **asymptotic form:** $\eta = \alpha \log|z| + o(1)$ **as** $|z| \to \infty$

- **At the apex,** $|z| \to 0$**,**

$$\eta = \begin{cases} 2l \log|z| + O(1) & \textbf{for} \quad |l| < \frac{1}{2} \\ \pm \log|z| + O\Big(\log(-\log|z|)\Big) & \textbf{for} \quad l = \pm\frac{1}{2} \end{cases}$$

Alday, Maldacena : $2\alpha = n - 2 = 2, 3\ldots,\ l = 0$



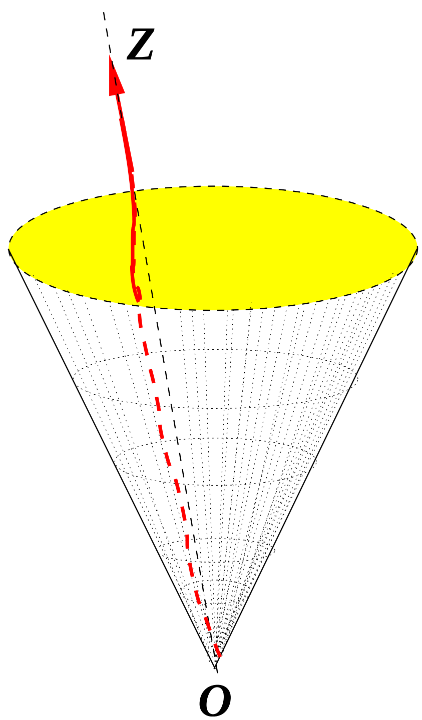
$$\partial_z \partial_{\bar{z}} \eta - \mathrm{e}^{2\eta} + p(z)\, \bar{p}(\bar{z})\, \mathrm{e}^{-2\eta} = 0 \Longrightarrow \qquad \partial_w \partial_{\bar{w}} \hat{\eta} - \mathrm{e}^{2\hat{\eta}} + \mathrm{e}^{-2\hat{\eta}} = 0$$

$$\hat{\eta}(w, \bar{w}) \to 0 \quad \textbf{as} \quad w \to \infty$$

MShG equation is integrable, and the associated flat $sl(2)$ connection reads

$$\begin{aligned} \boldsymbol{A}_z &= -\tfrac{1}{2}\,\partial_z\eta\;\sigma^3 + \mathrm{e}^{\theta}\;\Big[\sigma^+\;\mathrm{e}^{\eta} + \sigma^-\;p(z)\,\mathrm{e}^{-\eta}\Big] \\ \boldsymbol{A}_{\bar z} &= \tfrac{1}{2}\,\partial_{\bar z}\eta\;\sigma^3 + \mathrm{e}^{-\theta}\Big[\sigma^-\;\mathrm{e}^{\eta} + \sigma^+\;p(\bar z)\,\mathrm{e}^{-\eta}\Big] \end{aligned}$$

with the spectral parameter θ.

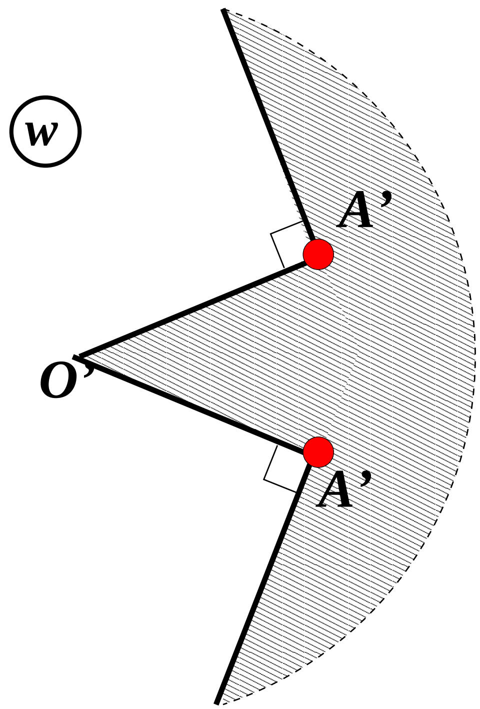


$$M_O^Z(\theta) = \mathcal{P}\exp\Big[\int_O^Z \mathrm{d}x^{\mu}\,A_{\mu}\Big]$$

Q: $\lim_{Z\to\infty} M_O^Z(\theta)$ does not exist. How to regularize it?

A: See **Faddeev-Takhtajan** book

Jost solutions



$$\hat{\Xi} \to \begin{pmatrix} 1 \\ -1 \end{pmatrix} \, \exp\Big(- w\, e^{\theta} - \bar{w}\, e^{-\theta} \Big) \quad w \to \infty$$

$$\Big[\partial_w + \tfrac{1}{2}\, \partial_w \hat{\eta}\; \sigma^3 - e^{\theta} \;\; \Big(\sigma^+ \, e^{\hat{\eta}} + \sigma^- e^{-\hat{\eta}} \Big) \Big] \; \hat{\Xi} = 0$$

$$\Big[\partial_{\bar{w}} - \tfrac{1}{2}\, \partial_{\bar{w}} \hat{\eta}\; \sigma^3 - e^{-\theta} \Big(\sigma^- \, e^{\hat{\eta}} + \sigma^+ e^{-\hat{\eta}} \Big) \Big] \; \hat{\Xi} = 0$$

$\hat{\Xi}$ **is an entire (!!!) function of** θ **and** $\sigma^3\, \hat{\Xi}(\theta + i\pi)$ **is another solution.**

$$M_O^{(\text{reg})}(\theta, \phi) = \frac{1}{2} \lim_{\substack{(w,\bar{w})\to O' \\ \arg(w)=\phi}} \Big(\hat{\Xi}(\theta),\;\; \sigma_3 \hat{\Xi}(\theta + i\pi) \Big) \in SL(2)$$

Regularized Monodromy Matrix

$$M_O^{(\mathrm{reg})}(\theta,\phi) = \frac{e^{-l(\theta+i\phi)\sigma_3}}{\sqrt{-2i\cos(\pi l)}} \begin{pmatrix} Q_+(\theta) & -ie^{-i\pi l}\, Q_+(\theta+i\pi) \\ Q_-(\theta) & ie^{i\pi l} \quad Q_-(\theta+i\pi) \end{pmatrix} \in SL(2)$$

Define the function $Q(\theta,k)$:

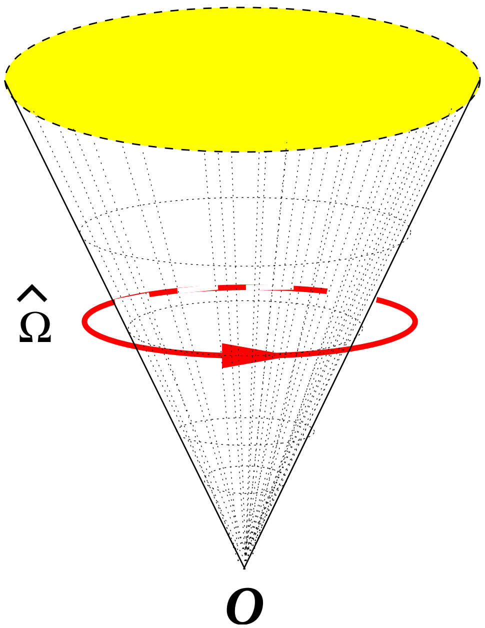
$$Q(\theta,k) := \begin{cases} Q_+(\theta)\Big|_{l=2k-\frac{1}{2}} & \text{for} \quad 0<k<\frac{1}{2} \\ Q_-(\theta)\Big|_{l=-2k-\frac{1}{2}} & \text{for} \quad -\frac{1}{2}<k<0 \end{cases}$$

Since

$$\lim_{l\to\pm\frac{1}{2}} Q_+(\theta) = \lim_{l\to\pm\frac{1}{2}} Q_-(\theta)$$

this definition extends to $k=0,\ k=\pm\frac{1}{2}$ by continuity, and then admits continuous periodic extension to all real k by

$$Q(\theta,k) = Q(\theta,k+1)$$



$$: \qquad \eta \to \eta, \qquad A_\mu(\theta)\mathrm{d}x^\mu \to A_\mu\Big(\theta + \frac{\mathrm{i}\pi}{\alpha}\Big)\ \mathrm{d}x^\mu$$

If $\Xi(z,\bar{z}|\theta)$ is the Jost solution of the linear problem

$$(\partial_z - A_z)\ \Xi = 0\ , \qquad (\partial_{\bar{z}} - A_{\bar{z}})\ \Xi = 0$$

then

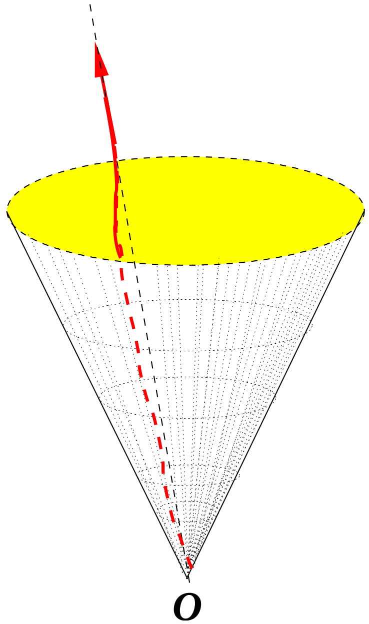
$$\hat{\Omega}^n[\Xi] := \Xi\Big(\mathrm{e}^{\frac{\mathrm{i}\pi n}{\alpha}}\, z\, , \mathrm{e}^{-\frac{\mathrm{i}\pi n}{\alpha}}\, \bar{z}\,\Big|\, \theta - \frac{\mathrm{i}\pi n}{\alpha}\Big)$$

are also solutions, but only Ξ and $\hat{\Omega}[\Xi]$ are linear independent. One can generate the connection coefficients $T_j(\theta)\ (j = \frac{1}{2}, 1, \frac{3}{2}\ldots)$:

$$\hat{\Omega}^n[\Xi] = -T_{\frac{n-2}{2}}\Big(\theta - \frac{\mathrm{i}\pi(\mathrm{n}+1)}{2\alpha}\Big)\ \Xi + T_{\frac{n-1}{2}}\Big(\theta - \frac{\mathrm{i}\pi n}{2\alpha}\Big)\ \hat{\Omega}[\Xi]$$

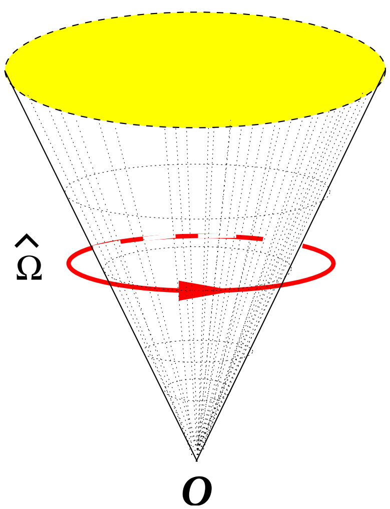
Connection coefficients for MShG

Q – function



$$M_O^{(\mathsf{reg})}(\theta,\phi) \propto \begin{pmatrix} Q(\theta,k) & \mathsf{e}^{-2\mathsf{i}\pi k}\, Q(\theta+\mathsf{i}\pi,k) \\ Q(\theta,-k) & \mathsf{e}^{2\mathsf{i}\pi k}\, Q(\theta+\mathsf{i}\pi,-k) \end{pmatrix}$$

T – functions



$$\hat{\Omega}^{2j+1} \quad : \qquad T_j(\theta) \qquad \left(j = 1,\, \tfrac{1}{2},\, 1, \ldots\right)$$

Quantum sine-Gordon model

$$\mathcal{L} = \frac{1}{16\pi}\ \Big[(\partial_t \varphi)^2 - (\partial_x \varphi)^2 \Big] + 2\mu \cos{(\beta\varphi)}$$

- **Dimensionless coupling constant** $\beta^2 < 1$**.**

- μ **sets the mass scale,** $\mu \sim [\,\mathbf{mass}\,]^{2-2\beta^2}$**.**

- **Finite-size geometry with periodic boundary**

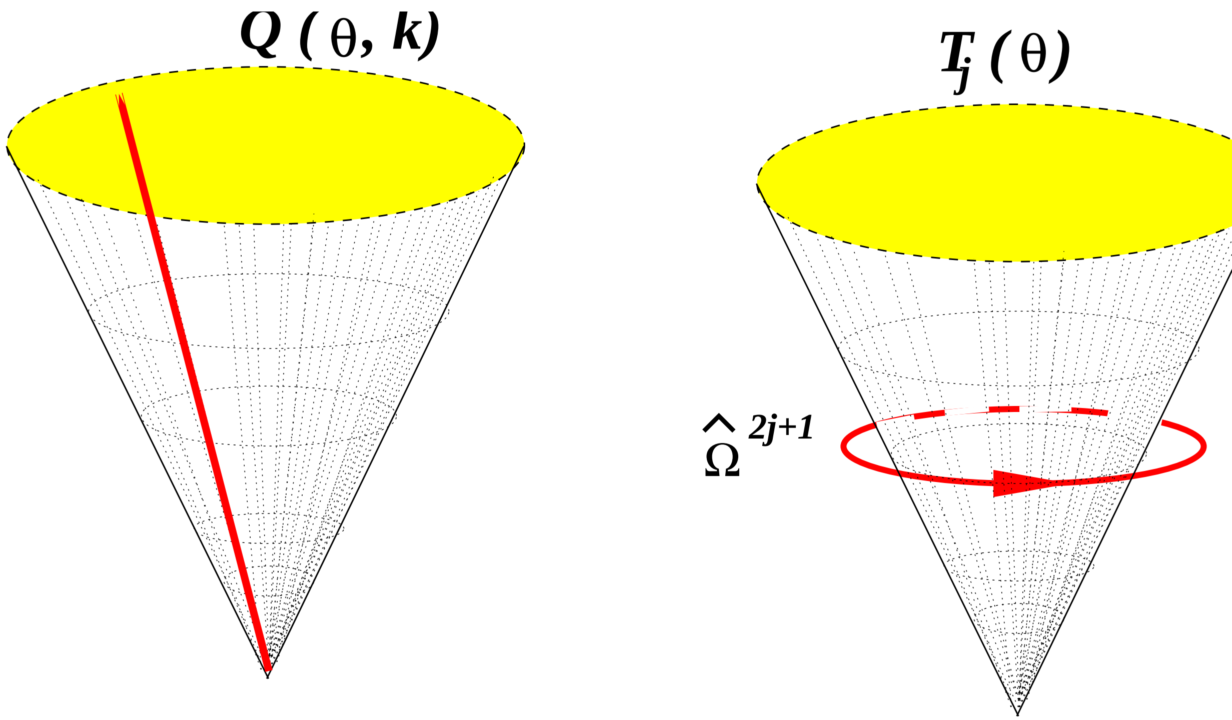
$$\varphi(x+R,t) = \varphi(x,t)$$

Due to the periodicity of the potential term $\cos(\beta\varphi)$**, the space of states** $\mathcal{H}$ **splits into orthogonal subspaces** $\mathcal{H}_k$**, characterized by the "quasi-momentum"** k**,**

$$\varphi \to \varphi + 2\pi/\beta\ : \qquad |\,\Psi_k\,\rangle\ \to\ \mathrm{e}^{2\pi \mathrm{i}\, k}\ |\,\Psi_k\,\rangle$$

for $|\,\Psi_k\,\rangle \in \mathcal{H}_k$**. We call** k**-vacuum the ground-state of the finite-size system in each in a given sector** $\mathcal{H}_k$**.**

Quantum sine-Gordon v.s. MShG



$Q(\theta,k)$ and $T_j(\theta)$ coincide with k-vacuum $(\in \mathcal{H}_k)$ eigenvalues of the $\mathbb{Q}(\theta)$ operator and the spin-j transfer-matrix $\mathbb{T}_j(\theta)$, respectively, of the quantum sine-Gordon model

$$k \; : \qquad l = 2|k| - \frac{1}{2}$$

is identified with the sine-Gordon quasi-momentum and

$$\alpha = \beta^{-2} - 1\,, \qquad\qquad s = \left(\frac{R}{\pi\beta^2}\right)^{\beta^2} \left[\frac{\mu\pi\Gamma(1-\beta^2)}{\Gamma(\beta^2)}\right]^{\frac{\beta^2}{2-2\beta^2}}$$

Integrable structures in quantum sine-Gordon

The quantum sine-Gordon field theory is integrable, in particular it has infinite set of

- **Local Integrals of Motion (IM)** $\mathbb{I}_{2n-1},\ \bar{\mathbb{I}}_{2n-1}$ **with the Lorentz spins** $\pm 1,\ \pm 3,\ \pm 5,\ \ldots$ **(Kulish, 1975)**

$$\mathbb{I}_{2n-1} = 2^{-4n}\int_0^R \frac{\mathrm{d}x}{2\pi}\ \Big[\big(\partial_x\varphi - \partial_t\varphi\big)^{2n} + \ldots\Big]$$
$$\bar{\mathbb{I}}_{2n-1} = 2^{-4n}\int_0^R \frac{\mathrm{d}x}{2\pi}\ \Big[\big(\partial_x\varphi + \partial_t\varphi\big)^{2n} + \ldots\Big]$$

- **Nonlocal IM** $\mathbb{G}_n,\ \bar{\mathbb{G}}_n$ **with fractional Lorentz spins**

  $\pm\frac{2n(1-\beta^2)}{\beta^2},\ n = 1, 2\ldots$ **(Leclair, 1990)**

These are mutually commuting families of operators

$$[\mathbb{I}_{2n-1}, \mathbb{I}_{2m-1}] = [\mathbb{I}_{2n-1}, \mathbb{G}_n] = [\mathbb{G}_n, \mathbb{G}_m] = \ldots = 0$$

$$\text{Hamiltonian} \ : \quad \mathbb{H} = \mathbb{I}_1 + \bar{\mathbb{I}}_1$$

T-operators
(Generating functions for local IM)

- XYZ-magnet (lattice counterpart of the quantum sine-Gordon) (Baxter (1972))

- QISM (Sklyanin, Takhtajan, Faddeev (1979))

- QFT approach (Bazhanov, Lukyanov, Zamomoldchikov (1996))

QFT approach

Quantum sine-Gordon = Gaussian CFT+perturbation

Gaussian CFT : $\partial_t^2\varphi - \partial_x^2\varphi = 0$
$\varphi(x+R,t) = \varphi(x,t)$

$$\varphi(x,t) \;=\; \frac{2}{\beta}\left[\phi(x-t) - \bar{\phi}(x+t)\right]$$
$$\phi(x_- + R) = \phi(x_-) + 2\pi p$$
$$\bar{\phi}(x_+ + R) = \phi(x_+) - 2\pi p$$

$$\mathcal{H}_{CFT} = \int_p \mathcal{F}_p \otimes \bar{\mathcal{F}}_{-p}$$

Here $\mathcal{F}_p$ ($\bar{\mathcal{F}}_{-p}$) **are Fock spaces (highest weight irreps) for** $\phi(x_-)$ ($\bar{\phi}(x_+)$)

$$\mathcal{H}_{sG} = \mathcal{H}_{CFT} = \int_{k\in\left(-\frac{1}{2},\frac{1}{2}\right]} \mathcal{H}_k$$

$$\mathcal{H}_k = \sum_{n=-\infty}^{\infty} \mathcal{F}_p \otimes \bar{\mathcal{F}}_{-p}\Big|_{p=(k+n)\beta^2}$$

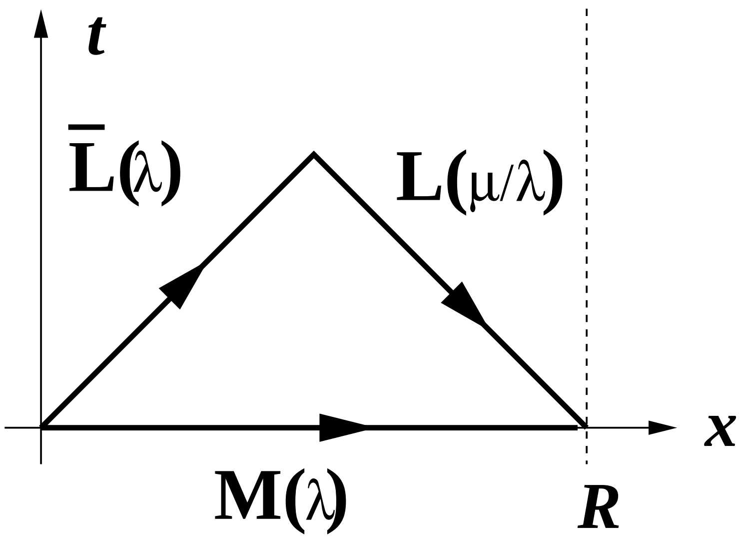
- $\mathbb{U}\ \mathcal{H}_k = \mathrm{e}^{2\mathrm{i}\pi k}\ \mathcal{H}_k$ **($\mathbb{U}$ is the Flouquet-Bloch operator)**

- **Hamiltonian** $\mathbb{H}$, **Parity** $\mathbb{P}$: $\mathcal{H}_k \to \mathcal{H}_k$

- **Charge conjugation** $\mathbb{C}$: $\mathcal{H}_k \to \mathcal{H}_{-k}$

$$\mathbb{C}\,\varphi(x,t)\,\mathbb{C} = -\varphi(x,t)$$
$$\mathbb{P}\,\varphi(x,t)\,\mathbb{P} = \varphi(-x,t)$$
$$\mathbb{U}\,\varphi(x,t)\,\mathbb{U}^{-1} = \varphi(x,t) + 2\pi/\beta$$

Quantum transfer-matrixes (T-operators)



$$\mathbb{M}(\lambda) = \mathbb{L}(\lambda)\bar{\mathbb{L}}(\mu/\lambda) \in \pi_j\Big[U_q\Big(sl(2)\Big)\Big]$$
$$(q = \mathrm{e}^{\mathrm{i}\pi\beta^2})$$

$$\mathbb{T}_j(\theta) = \mathbf{Tr}_{\pi_j}\Big[\mathbb{M}(\lambda)\Big] = \sum_{n=-\infty}^{\infty} t_n^{(j)}\ \lambda^{2n}\ , \qquad \lambda = const\ \mathrm{e}^{(1-\beta^2)\theta}$$

Basic properties of T-operators

- **Mutual commutativity** $[\,\mathbb{T}_j(\theta),\, \mathbb{T}_{j'}(\theta')\,] = 0$

- $\mathbb{U}$ **and** $\mathbb{C}$ **invariance:** $[\,\mathbb{T}_j(\theta)\,,\, \mathbb{U}\,] = [\,\mathbb{T}_j(\theta)\,,\, \mathbb{C}\,] = 0$

- **Parity transformation** $\mathbb{P}\,\mathbb{T}_j(\theta)\,\mathbb{P} = \mathbb{T}_j(-\theta)$

- **Hermiticity:** $\mathbb{T}_j^\dagger(\theta) = \mathbb{T}_j(\theta^*)$

- **Periodicity:** $\mathbb{T}_j\Big(\theta + \frac{\mathrm{i}\pi(\alpha+1)}{\alpha}\Big) = \mathbb{T}_j(\theta) \quad \Big(\mathbb{T}_j = \sum_{n=-\infty}^{\infty} t_n^{(j)}\ \lambda^{2n}\Big)$

- **Fusion** $(\alpha = \beta^{-2} - 1)$

$$\mathbb{T}_{\frac{1}{2}}(\theta)\ \mathbb{T}_j\Big(\theta + \frac{\mathrm{i}\pi(2\mathrm{j}+1)}{2\alpha}\Big) = \mathbb{T}_{j-\frac{1}{2}}\Big(\theta + \frac{\mathrm{i}\pi(2\mathrm{j}+2)}{2\alpha}\Big) + \mathbb{T}_{j+\frac{1}{2}}\Big(\theta + \frac{2\mathrm{ij}\pi}{2\alpha}\Big)$$

- **Asymptotic at real θ:**

$$\log \mathbb{T}_{\frac{1}{2}}(\theta) \sim \sum_{n=0}^{\infty} \; C_n \; \begin{cases} \mathbb{I}_{2n-1}\ \mathrm{e}^{-(2\mathrm{n}-1)\theta} & \text{as} \quad \theta \to +\infty \\ \bar{\mathbb{I}}_{2n-1}\ \mathrm{e}^{(2\mathrm{n}-1)\theta} & \text{as} \quad \theta \to -\infty \end{cases}$$

Here $\mathbb{I}_{-1} = \bar{\mathbb{I}}_{-1} = \frac{R}{2\pi}$ **is a** c**-number, while** $\mathbb{I}_{2n-1}$ $(n = 1, 2\ldots)$ **are local IM**

$$\mathbb{I}_{2n-1} \;=\; 2^{-4n} \int_0^R \frac{\mathrm{d}x}{2\pi} \; \Big[\Big(\partial_x\varphi - \partial_t\varphi\Big)^{2n} + \ldots \Big]$$

$$\bar{\mathbb{I}}_{2n-1} \;=\; 2^{-4n} \int_0^R \frac{\mathrm{d}x}{2\pi} \; \Big[\Big(\partial_x\varphi + \partial_t\varphi\Big)^{2n} + \ldots \Big]$$

and C_n **are known constants.**

$\mathbb{Q}$-operator

- **Commutativity:** $[\,\mathbb{T}_j(\theta),\, \mathbb{Q}(\theta')\,] = [\,\mathbb{Q}(\theta),\, \mathbb{Q}(\theta')\,] = 0$

- $\mathbb{U}$ **invariance:** $[\,\mathbb{Q}(\theta)\,,\,\mathbb{U}\,]$

- **Baxter Equation:** $\mathbb{T}_{\frac{1}{2}}(\theta)\ \mathbb{Q}(\theta) = \mathbb{Q}\Big(\theta + \frac{\mathrm{i}\pi}{\alpha}\Big) + \mathbb{Q}\Big(\theta - \frac{\mathrm{i}\pi}{\alpha}\Big)$

- **Quasiperiodicity:** $\mathbb{Q}\Big(\theta + \frac{\mathrm{i}\pi(\alpha+1)}{\alpha}\Big) = \mathbb{U}\ \mathbb{Q}(\theta)$

- $\mathbb{Q}(\theta)$ **and** $\mathbb{C}\,\mathbb{Q}(\theta)\,\mathbb{C}$ **satisfy Quantum Wronskian relation:**

  $\mathbb{Q}\Big(\theta + \frac{\mathrm{i}\pi}{2\alpha}\Big)\ \mathbb{C}\,\mathbb{Q}\Big(\theta - \frac{\mathrm{i}\pi}{2\alpha}\Big)\,\mathbb{C} - \mathbb{Q}\Big(\theta - \frac{\mathrm{i}\pi}{2\alpha}\Big)\ \mathbb{C}\,\mathbb{Q}\Big(\theta + \frac{\mathrm{i}\pi}{2\alpha}\Big)\,\mathbb{C} = \mathbb{U}^{-1} - \mathbb{U}$

- **Parity transformation** $\mathbb{P}\,\mathbb{Q}(\theta)\,\mathbb{P} = \mathbb{C}\,\mathbb{Q}(-\theta)\,\mathbb{C}$

- **Hermiticity:** $\mathbb{Q}^\dagger(\theta) = \mathbb{Q}(\theta^*)$

$\mathbb{Q}$-operator
(Generating functions for all IM)

- **Large-θ asymptotic:**

$$\log \mathbb{Q}\Big(\theta + \frac{\mathrm{i}\pi(\alpha+1)}{2\alpha}\Big) \sim \sum_{n=0}^{\infty}$$

$$\begin{cases} \tilde{C}_n \ \mathbb{I}_{2n-1} \ \mathrm{e}^{-(2\mathrm{n}-1)\theta} + \mathbb{G}_{\mathrm{n}} \ \mathrm{e}^{-2\mathrm{n}\alpha\theta} & \text{as} \quad \theta \to +\infty \\ \tilde{C}_n \ \bar{\mathbb{I}}_{2n-1} \ \mathrm{e}^{+(2\mathrm{n}-1)\theta} + \bar{\mathbb{G}}_{\mathrm{n}} \ \mathrm{e}^{+2\mathrm{n}\alpha\theta} & \text{as} \quad \theta \to -\infty \end{cases}$$

Here $\mathbb{I}_{-1} = \bar{\mathbb{I}}_{-1} = \frac{R}{2\pi}$ **is a** c**-number,** $\mathbb{I}_{2n-1}$ $(n = 1, 2\dots)$ **are local IM**

$$\mathbb{I}_{2n-1} \ = \ 2^{-4n} \int_0^R \frac{\mathrm{d}x}{2\pi} \ \Big[\Big(\partial_x\varphi - \partial_t\varphi\Big)^{2n} + \dots\Big]$$

$$\bar{\mathbb{I}}_{2n-1} \ = \ 2^{-4n} \int_0^R \frac{\mathrm{d}x}{2\pi} \ \Big[\Big(\partial_x\varphi + \partial_t\varphi\Big)^{2n} + \dots\Big]$$

and $\tilde{C}_n$ **are known constants.**

$\mathbb{G}_n$, $\bar{\mathbb{G}}_n$ **are nonlocal IM with fractional Lorentz spins**

$\pm 2n\alpha,\ n = 0, 1\dots$ **(Leclair, 1990)**

Q- and T-functions

$$\mathcal{H}_k = \sum_{n=-\infty}^{\infty} \oplus_{p=2(k+n)\beta^2} \mathcal{F}_p \otimes \bar{\mathcal{F}}_{-p}$$

$$\mathbb{U}\ \mathcal{H}_k = \mathrm{e}^{2\mathrm{i}\pi k}\ \mathcal{H}_k$$

k**-vacuum states:** $|\,k\,\rangle \in \mathcal{H}_k \qquad \mathbb{P}\ |\,k\,\rangle = |\,k\,\rangle$

$$\mathbb{Q}(\theta)\ |\,k\,\rangle = Q(\theta,k)\,|\,k\,\rangle\ , \qquad \mathbb{T}_j(\theta)\ |\,k\,\rangle = T_j(\theta,k)\,|\,k\,\rangle$$

Characteristic properties of Q-function

- **Quasiperiodicity:** $Q\Big(\theta + \frac{\mathrm{i}\pi(\alpha+1)}{\alpha}, k\Big) = \mathrm{e}^{2\pi\mathrm{i}k}\ Q(\theta, k)$

- $Q(\theta, k+1) = Q(\theta, k)$

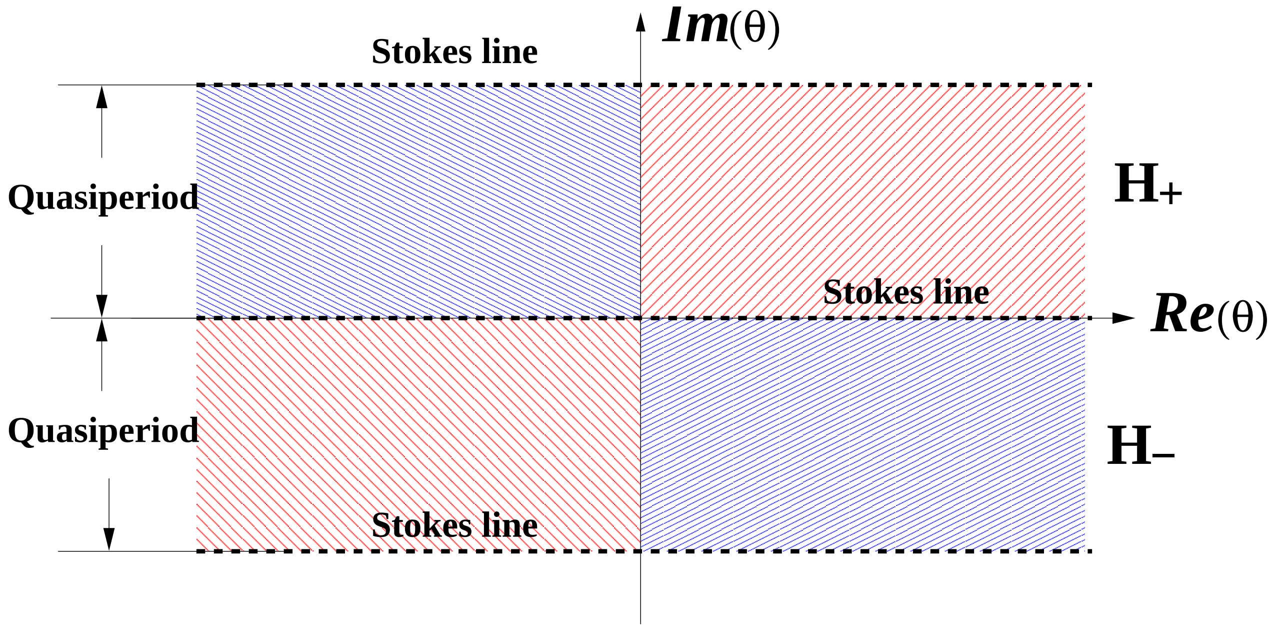
- **$\mathbb{P}$-invariance:** $Q(-\theta, k) = Q(\theta, -k)$

- **Conjugation:** $Q^*(\theta, k) = Q(\theta^*, k)$

- **Quantum Wronskian relation:**

$$Q\Big(\theta + \tfrac{\mathrm{i}\pi}{2\alpha}, k\Big)\ Q\Big(\theta - \tfrac{\mathrm{i}\pi}{2\alpha}, -k\Big) - Q\Big(\theta - \tfrac{\mathrm{i}\pi}{2\alpha}, k\Big)\ Q\Big(\theta + \tfrac{\mathrm{i}\pi}{2\alpha}, -k\Big) = -2\mathrm{i}\ \sin(2\pi k)$$

- **Analiticity:** $Q(\theta, k)$ **is an entire function of** θ



- **Leading large θ-asymptotic**

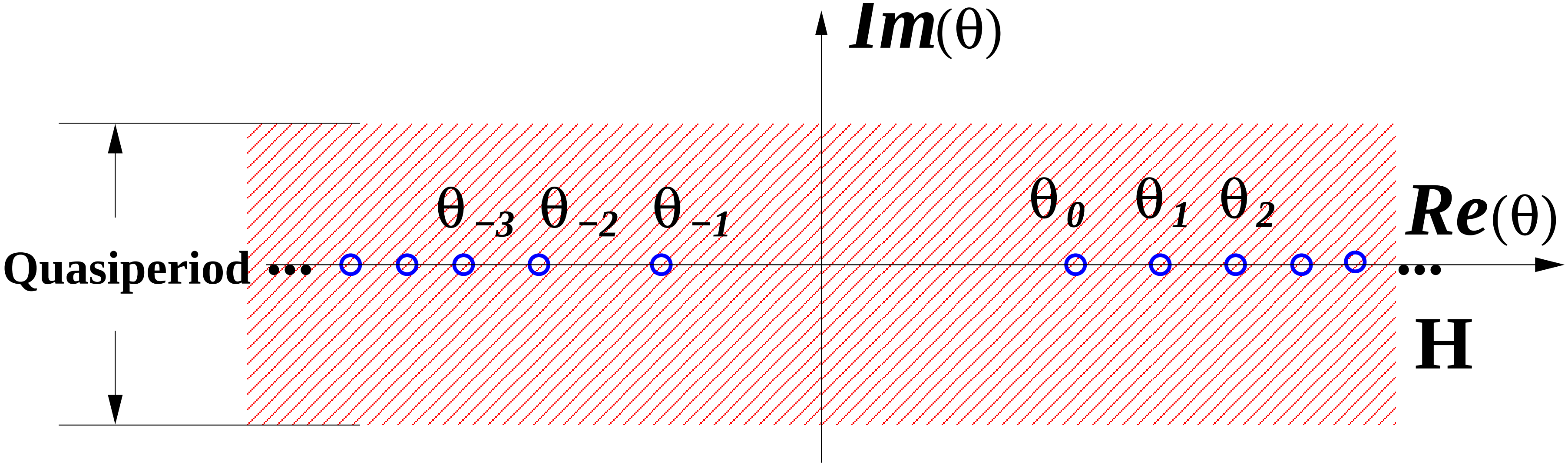
$$Q \;\to\; \mathrm{e}^{\pm \mathrm{i}\pi k}\, \mathfrak{S}^{\frac{1}{2}} \quad \exp\Big[C\ \mathrm{e}^{\theta \mp \frac{\mathrm{i}\pi(1+\alpha)}{2\alpha}}\Big] \qquad \Big(\theta\in \mathsf{H}_\pm\,,\ \ \Re\mathrm{e}\,(\theta)\to +\infty\Big)$$

$$Q \;\to\; \mathrm{e}^{\pm \mathrm{i}\pi k}\, \mathfrak{S}^{-\frac{1}{2}} \ \exp\Big[C\ \mathrm{e}^{-\theta \pm \frac{\mathrm{i}\pi(1+\alpha)}{2\alpha}}\Big] \quad \Big(\theta\in \mathsf{H}_\pm\,,\ \ \Re\mathrm{e}\,(\theta)\to -\infty\Big)$$

Here $C = \frac{MR}{4\cos(\frac{\pi}{2\alpha})}$ **and** $\mathfrak{S}=\mathfrak{S}(k)$**- vacuum eigenvalue of the spin-0 unlocal IM ("reflection** S**-martix"):**

$$\mathfrak{S}(k)\,\mathfrak{S}(-k) = 1\ , \qquad \mathfrak{S}(k+1)=\mathfrak{S}(k)$$

- Zeroes



For any real k, all the zeros of $Q(\theta, k)$ in the strip H are real, simple, and accumulate towards $\theta \to \pm\infty$. Let

$$\epsilon(\theta) = \mathrm{i}\ \log\left[\frac{Q\left(\theta + \frac{\mathrm{i}\pi}{\alpha}, k\right)}{Q\left(\theta - \frac{\mathrm{i}\pi}{\alpha}, k\right)}\right] ,$$

then the zeros θ_n can be labeled by consecutive integers $n = 0, \pm 1, \pm 2, ...$, so that $\theta_n < \theta_{n+1}$, and

$$\epsilon(\theta_n) = \pi\ (2n + 1)$$

The quasiperiodic entire function $Q(\theta, k)$ is completely determined by its zeros θ_n in the strip H and the large-θ asymptotic. On the other hand, the positions of the zeros θ_n are restricted by the equation

$$\epsilon(\theta_n) = \mathrm{i}\ \log \left[\frac{Q\left(\theta_n + \frac{\mathrm{i}\pi}{\alpha}, k\right)}{Q\left(\theta_n - \frac{\mathrm{i}\pi}{\alpha}, k\right)} \right] = \pi\ (2n+1)$$

Mathematically, the problem of reconstructing the function $Q(\theta, k)$ from this data has emerged long ago in the context of the analytic Bethe ansatz

- Baxter (1972)
- Sklyanin, Takhtajan, Faddeev (1979)
- Reshetikhin (1983)

For the sine-Gordon model, the problem was solved by Destri and De Vega (1992), who have reduced it to a single complex integral equation, the celebrated DDV equation.

DDV equation

$$2\int_{-\infty}^{\infty} \mathrm{d}\theta' \, \Im m \left[\log\left(1 + \mathrm{e}^{-\mathrm{i}\epsilon(\theta' - \mathrm{i}0)} \right) \right] G(\theta - \theta') = r \sinh(\theta) - 2\pi k - \epsilon(\theta)$$

where

$$G(\theta) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\nu}{2\pi} \frac{\mathrm{e}^{\mathrm{i}\theta\nu} \sinh\left(\frac{\pi\nu(1-\alpha)}{2\alpha}\right)}{2\cosh\left(\frac{\pi\nu}{2}\right) \sinh\left(\frac{\pi\nu}{2\alpha}\right)}$$

and $r = MR$. **Then**

$$\epsilon(\theta) = \mathrm{i} \; \log\left[\frac{Q\left(\theta + \frac{\mathrm{i}\pi}{\alpha}, k\right)}{Q\left(\theta - \frac{\mathrm{i}\pi}{\alpha}, k\right)} \right]$$

can be used to reconstruct the Q**-function.**

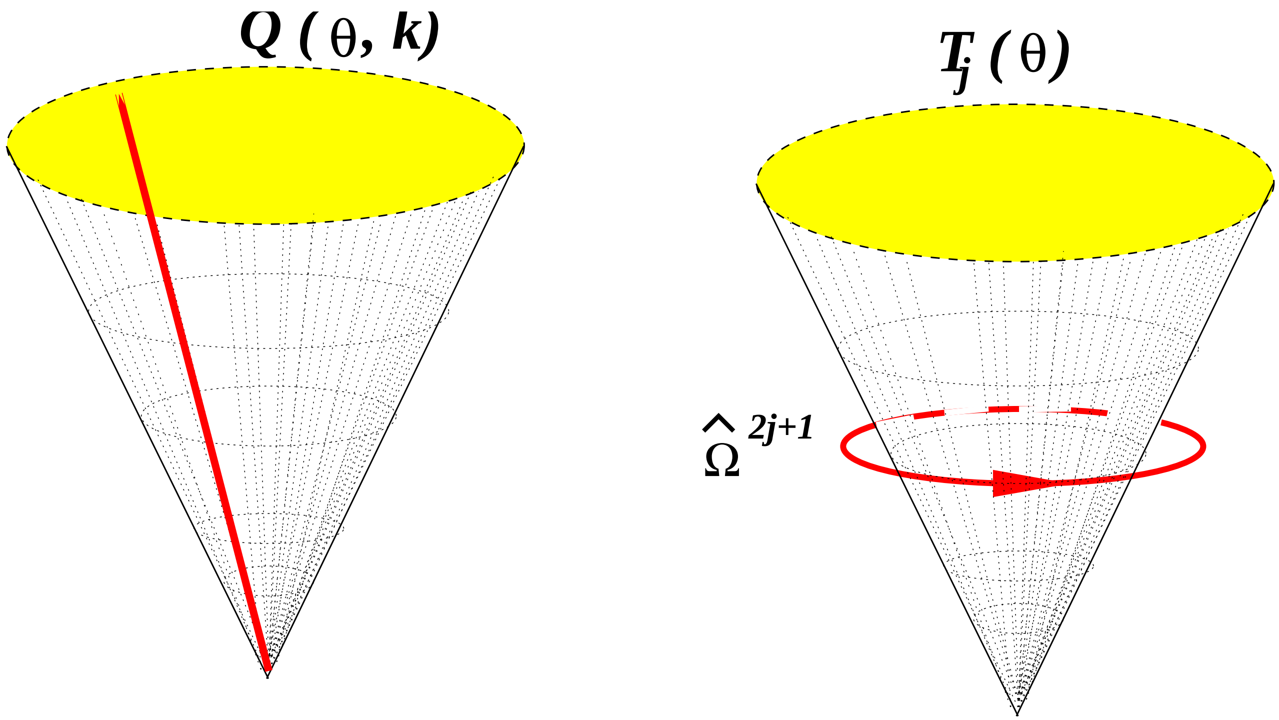
For $\alpha = 1$ **("free fermions"),** $G(\theta) = 0$**:**

$$\epsilon(\theta)|_{\alpha=1} = r \sinh(\theta) - 2\pi k$$

This corresponds to the MShG with $p(z) = z^2 - s^2$ $(r = \pi s^2)$

Gaiotto, Moore, Neitzke (2008)

Quantum sine-Gordon v.s. MShG



$Q(\theta, k)$ and $T_j(\theta)$ coincide with k-vacuum $(\in \mathcal{H}_k)$ eigenvalues of the $\mathbb{Q}(\theta)$ operator and the spin-j transfer-matrix $\mathbb{T}_j(\theta)$, respectevely, of the quantum sine-Gordon model

$$k \quad : \qquad l = 2|k| - \frac{1}{2}$$

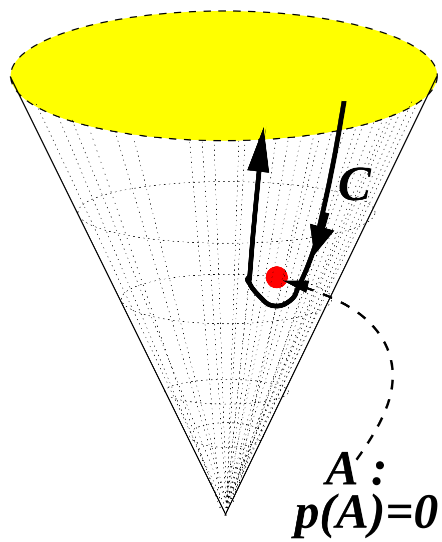
is identified with the sine-Gordon quasi-momentum and

$$\alpha = \beta^{-2} - 1\,, \qquad\qquad s = \left(\frac{R}{\pi\beta^2}\right)^{\beta^2} \left[\frac{\mu\pi\Gamma(1-\beta^2)}{\Gamma(\beta^2)}\right]^{\frac{\beta^2}{2-2\beta^2}}$$

Local IM

The large-θ asymptotic expansions for $Q(\theta)$ and $T_j(\theta)$ can be obtained directly from the WKB expansion for the linear problem.

Local IM for classical MShG $=$ k-vacuum eigenvalues of local quantum IM.



$$E = -\frac{\pi(2\alpha-1)}{12R(\alpha+1)} + \frac{M}{2} \int_C \left[\mathrm{d}z \, \frac{u}{2\sqrt{p}} + \mathrm{d}\bar{z} \, \sqrt{\bar{p}} \, \left(\sqrt{p\bar{p}} \, \mathrm{e}^{-2\eta} - 1 \right) \right]$$

$$u = (\partial_z \eta)^2 - \partial_z^2 \eta$$

Conclusion

- We discussed the relation between the classical MShG equation and quantum sine-Gordon model. It generalizes the relation (Dorey, Toteo (1998)) between ordinary differential equations (A. Voros (1992)) and integrable structures of Conformal Field Theories to the massive case.

- Can the excited-state eigenvalues be also related to integrable classical equations? (In the massless case: BLZ (2001))

- It seems that the relation between classical equations and quantum theories is a quite general feature of 2D integrable QFT which deserves to be explored in whole details.