Fluctuations of the current in the Asymmetric Simple Exclusion Process

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I. The Asymmetric Simple Exclusion Process

II. Bethe Ansatz for the fluctuations of the current

III. Exact solution of Baxter's equation

IV. Tree structures for the cumulants of the current

Introduction

Equilibrium systems: microscopic description by the Boltzmann – Gibbs measure.

$$P_{\text{eq}}(\mathcal{C}) = \frac{1}{Z} e^{-E(\mathcal{C})/kT}$$

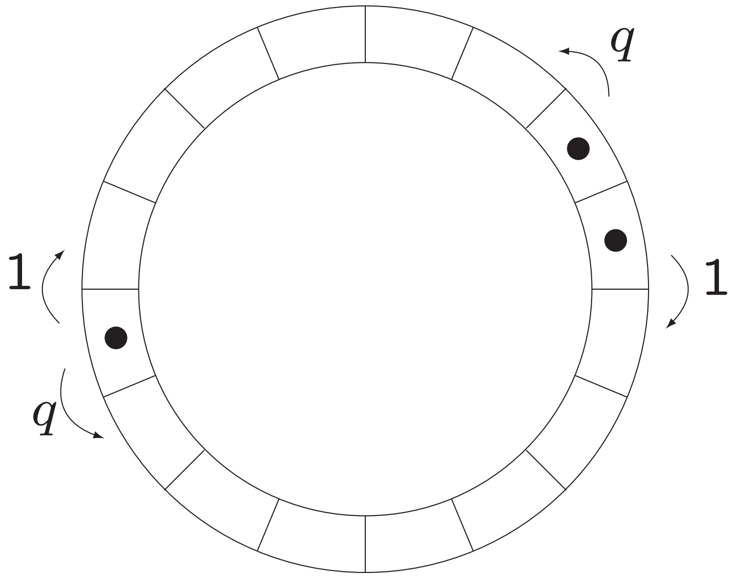
Systems far from equilibrium: no general theory for the probability to observe the system in a given microstate, even in a stationary state where $P(\mathcal{C})$ does not depend on time.

$$P_{\text{stat}}(\mathcal{C}) = ?$$

The study of exactly solvable models helps to understand out of equilibrium phenomena.

$\hookrightarrow$ Asymmetric Simple Exclusion Process

The Asymmetric Simple Exclusion Process (ASEP)



L sites, n classical particles

Exclusion constraint: at most one particle per site

$\Omega = \binom{L}{n}$ configurations

hopping rates 1 and q

Variants: open model, several species of particles, . . .

Out of equilibrium stochastic model: stationary currents breaking detailed balance if $q \neq 1$.

Model for physical systems: cellular molecular motors, hopping conductivity, traffic flow, . . .

Quantum integrable model: exact calculations possible.

Time evolution of the probability

Probability $P_t(\mathcal{C})$ to observe the system in configuration $\mathcal{C}$ at time t.

Time evolution of $P_t(\mathcal{C})$ given by the master equation

$$\frac{dP_t(\mathcal{C})}{dt} = \sum_{\mathcal{C}' \neq \mathcal{C}} \left[w_{\mathcal{C} \leftarrow \mathcal{C}'} P_t(\mathcal{C}') - w_{\mathcal{C}' \leftarrow \mathcal{C}} P_t(\mathcal{C}) \right]$$

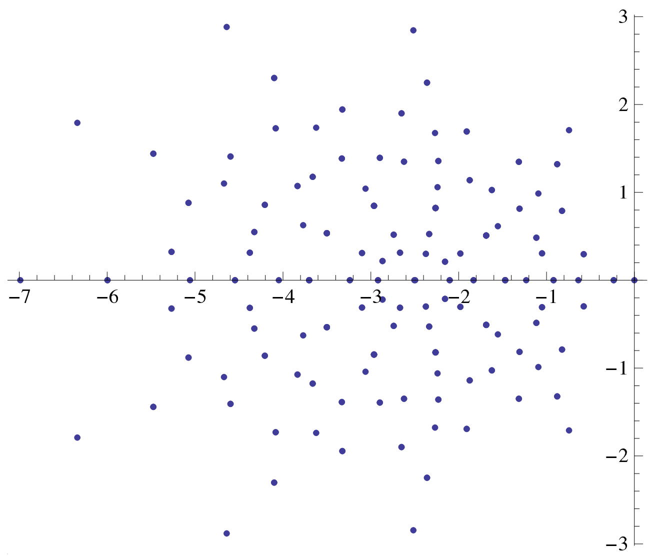
Matrix form (M Markov matrix):

$$\frac{d|P_t\rangle}{dt} = M|P_t\rangle \qquad \Rightarrow \qquad |P_t\rangle = e^{Mt}|P_0\rangle$$

M has one eigenvalue equal to 0.

All the other eigenvalues have a strictly negative real part.

M not symmetric ($q \neq 1$)
$\Rightarrow$ complex spectrum.



$L = 10$
$n = 5$

$q = 0$

Total current

Let Y_t be the total distance covered by all the particles (integrated current) between time 0 and time t.

$$\frac{dP_t(\mathcal{C},Y)}{dt} = \sum_{\mathcal{C}'\neq\mathcal{C}} \left[w^{(+)}_{\mathcal{C}\leftarrow\mathcal{C}'} P_t(\mathcal{C}',Y-1) + w^{(-)}_{\mathcal{C}\leftarrow\mathcal{C}'} P_t(\mathcal{C}',Y+1) - w_{\mathcal{C}'\leftarrow\mathcal{C}} P_t(\mathcal{C},Y) \right]$$

$P_t(\mathcal{C},Y)$ coupled for different values of Y

Introduction of a parameter γ, fugacity associated to particle hopping:

$$F_t(\mathcal{C},\gamma) = \sum_{Y=-\infty}^{\infty} e^{\gamma Y} P_t(\mathcal{C},Y) = \langle e^{\gamma Y_t} \rangle_{\mathcal{C}}$$

$\Rightarrow$ deformation of the master equation:

$$\frac{dF_t(\mathcal{C},\gamma)}{dt} = \sum_{\mathcal{C}'\neq\mathcal{C}} \left[e^{\gamma} w^{(+)}_{\mathcal{C}\leftarrow\mathcal{C}'} F_t(\mathcal{C}',\gamma) + e^{-\gamma} w^{(-)}_{\mathcal{C}\leftarrow\mathcal{C}'} F_t(\mathcal{C}',\gamma) - w_{\mathcal{C}'\leftarrow\mathcal{C}} F_t(\mathcal{C},\gamma) \right]$$

$F_t(\mathcal{C},\gamma)$ decoupled for different values of γ.

Fluctuations of the current

Introduce the deformed Markov matrix $M(\gamma)$

$$\frac{d|F_t\rangle}{dt} = M(\gamma)|F_t\rangle \qquad \Rightarrow \qquad |F_t\rangle = e^{M(\gamma)t}|F_0\rangle$$

In the long time limit

$$\langle e^{\gamma Y_t}\rangle \sim e^{E(\gamma)t}$$

with $E(\gamma)$ the eigenvalue of $M(\gamma)$ with largest real part.

$E(\gamma)$ is the generating function of the cumulants of the stationary current:

$$E(\gamma) = J\gamma + \frac{D}{2!}\gamma^2 + \frac{E_3}{3!}\gamma^3 + \frac{E_4}{4!}\gamma^4 + \dots$$

$$J = \lim_{t\to\infty} \frac{\langle Y_t\rangle}{t} \qquad D = \lim_{t\to\infty} \frac{\langle (Y_t - \langle Y_t\rangle)^2\rangle}{t}$$

$$E_3 = \lim_{t\to\infty} \frac{\langle (Y_t - \langle Y_t\rangle)^3\rangle}{t} \qquad E_4 = \lim_{t\to\infty} \frac{\langle (Y_t - \langle Y_t\rangle)^4\rangle - 3\langle (Y_t - \langle Y_t\rangle)^2\rangle^2}{t}$$

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Calculation of $E(\gamma)$: Bethe Ansatz

The matrix $M(\gamma)$ is related through a similarity transformation to the (non hermitian) Hamiltonian of the XXZ spin chain $\left(\Delta \equiv \frac{1}{2}\left(\sqrt{q} + \frac{1}{\sqrt{q}}\right) \geq 1\right)$

$$M(\gamma) \sim H_{XXZ} = -\frac{1}{2} \sum_{i=1}^{L} \left(S_i^{(x)} S_{i+1}^{(x)} + S_i^{(y)} S_{i+1}^{(y)} + \Delta S_i^{(z)} S_{i+1}^{(z)} \right)$$

with "twisted" boundary conditions:

$$S_{L+1}^{(+)} = \left(\sqrt{\frac{q}{e^{2\gamma}}}\right)^{-L} S_1^{(+)} \qquad S_{L+1}^{(-)} = \left(\sqrt{\frac{q}{e^{2\gamma}}}\right)^{L} S_1^{(-)} \qquad S_{L+1}^{(z)} = S_1^{(z)}$$

$M(\gamma)$ is also related to the transfer matrix of the six vertex model with nonzero external fields.

$M(\gamma)$ is thus diagonalizable using Bethe Ansatz

Bethe equations

Eigenvalues of $M(\gamma)$:

$$E = \sum_{j=1}^{n} \left(\frac{e^{\gamma}}{z_j} + q e^{-\gamma} z_j - (1+q) \right)$$

Bethe equations:

$$z_i^L = (-1)^{n-1} \prod_{j=1}^{n} \frac{1-(1+q)e^{-\gamma} z_i + q e^{-2\gamma} z_i z_j}{1-(1+q)e^{-\gamma} z_j + q e^{-2\gamma} z_i z_j}$$

Among all the solutions of the Bethe equations, we are interested in the one corresponding to the largest eigenvalue of $M(\gamma)$ (stationary state).

Selection of the solution corresponding to the largest eigenvalue:

$$\lim_{\gamma \to 0} z_i(\gamma) = 1$$

For this solution of the Bethe equations

$$\prod_{i=1}^{n} z_i = 1 \qquad \text{and} \qquad \lim_{\gamma \to 0} E(\gamma) = 0$$

Totally asymmetric model ($q = 0$)

For the totally asymmetric model (TASEP, all the particles hop in the same direction), the Bethe equations "decouple":

$$(z_i - e^{\gamma})^n z_i^{-L} = (-1)^{n-1} \prod_{j=1}^{n} (z_j - e^{\gamma})$$

The second member of the equation does not depend on i: it depends symmetrically on all the z_j.

Parametric expression for the generating function of the cumulants of the current (Derrida & Lebowitz, PRL **80**, 1998)

$$E(\gamma) = -\frac{n(L-n)}{L} \sum_{k=1}^{\infty} \binom{kL}{kn} \frac{B^k}{kL-1}$$

$$\gamma = -\frac{1}{L} \sum_{k=1}^{\infty} \binom{kL}{kn} \frac{B^k}{k}$$

Finite size system

$$\frac{E(\gamma)-\rho(1-\rho)L\gamma}{\sqrt{\rho(1-\rho)}} \sim -\frac{\mathrm{Li}_{5/2}(C)}{\sqrt{2\pi L^3}}$$

$$L^{3/2}\gamma \sim -\frac{\mathrm{Li}_{3/2}(C)}{\sqrt{2\pi\rho(1-\rho)}}$$

$L \to \infty$, $\gamma \sim L^{-3/2}$, $\frac{n}{L} = \rho$

Partially asymmetric model ($0 < q < 1$)

If $q \neq 0$, the Bethe equations do not decouple anymore

$$z_i^L = (-1)^{n-1} \prod_{j=1}^{n} \frac{1-(1+q)e^{-\gamma} z_i + q e^{-2\gamma} z_i z_j}{1-(1+q)e^{-\gamma} z_j + q e^{-2\gamma} z_i z_j}$$

Calculation of the cumulants of the current ?
$\hookrightarrow$ rewrite the Bethe equations as a functional equation (Baxter's equation).

Change of variables in the Bethe equations

$$z_i = e^{\gamma} \frac{1-y_i}{1-qy_i} \qquad \Rightarrow \qquad e^{L\gamma}(1-y_i)^L Q(qy_i) + q^n (1-qy_i)^L Q(y_i/q) = 0$$

where the polynomial Q defined by

$$Q(t) = \prod_{j=1}^{n} (t - y_j)$$

is the polynomial whose zeros are the y_j.

Baxter's (scalar) TQ equation

Functional equation:

$$\boxed{Q(t)T(t) = e^{L\gamma}(1-t)^L Q(qt) + q^n (1-qt)^L Q(t/q)}$$

Baxter's (scalar) TQ equation

Two unknown polynomials: Q of degree n and T of degree L

Equivalent to the Bethe equations: the Bethe roots are the zeros of Q.

Choice of the eigenstate corresponding to the largest eigenvalue:

$$Q(t) = t^n + \mathcal{O}(\gamma) \quad \Rightarrow \quad \text{perturbative expansion in } \gamma$$

Corresponding eigenvalue

$$E(\gamma) = (1-q)\left(\frac{Q'(1)}{Q(1)} - \frac{1}{q}\frac{Q'(1/q)}{Q(1/q)}\right)$$

First cumulants of the current

Mean value of the current:

$$J = (1-q)\frac{n(L-n)}{L-1}$$

Diffusion constant:

$$\frac{(L-1)D}{(1-q)L} = \sum_{i\in\mathbb{Z}} i^2 \frac{\binom{L}{n+i}\binom{L}{n-i}}{\binom{L}{n}^2} \frac{1+q^{|i|}}{1-q^{|i|}}$$

Third cumulant of the current ⇒ non gaussianity:

$$\begin{aligned}\frac{(L-1)E_3}{(1-q)L^2} =& \frac{1}{6}\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\left(i^2+ij+j^2\right)\frac{\binom{L}{n+i}\binom{L}{n+j}\binom{L}{n-i-j}}{\binom{L}{n}^3} \\ &- \frac{3}{2}\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\left(i^2+ij+j^2\right)\frac{\binom{L}{n+i}\binom{L}{n+j}\binom{L}{n-i-j}}{\binom{L}{n}^3}\frac{1+q^{|i|}}{1-q^{|i|}}\frac{1+q^{|j|}}{1-q^{|j|}} \\ &+ \frac{3}{2}\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\left(i^2+j^2\right)\frac{\binom{L}{n+i}\binom{L}{n-i}\binom{L}{n+j}\binom{L}{n-j}}{\binom{L}{n}^4}\frac{1+q^{|i|}}{1-q^{|i|}}\frac{1+q^{|j|}}{1-q^{|j|}}\end{aligned}$$

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Quantum Wronskian

Higher cumulants of the current ?

Using Baxter's TQ equation, another functional equation can be written (Pronko & Stroganov, J. Phys. A **32**, 1999): the "Quantum Wronskian"

$$\boxed{(1 - q^n e^{-L\gamma}) Q(0)(1-t)^L = Q(t)P(t/q) - q^n e^{-L\gamma} Q(t/q)P(t)}$$

Two unknown polynomials: Q of degree n and P of degree $L - n$

Remark: P and T are also solution of Baxter's equation "beyond the equator" $(n \to L - n)$

$$P(t)T(t) = q^n (1-t)^L P(qt) + e^{L\gamma}(1-qt)^L P(t/q)$$

The equation for P and Q still depends on two unknown polynomials. It can be rewritten as an equation for only one unknown function.

Functions α and β

We define the two functions

$$\alpha(t) \equiv \log\left(\frac{q^n Q(t/q)}{Q(t)}\right) \qquad \text{and} \qquad \beta(t) \equiv \log\left(\frac{P(t/q)}{P(t)}\right)$$

The key point will be that $\alpha(t)$ has only negative powers in t while $\beta(t)$ has only positive powers in t, which can be understood either:

- as a formal series in γ: at each order in γ, $\alpha(t)$ is a polynomial in $1/t$ while $\beta(t)$ is a polynomial in t
- for finite $\gamma > 0$, as a Laurent series in t for t inside an annulus in the complex plane

With this property, the functional equation for P and Q can be rewritten so that it depends on P and Q only through the function $\alpha - \beta$.

Then, the equation for $\alpha(t) - \beta(t)$ can be solved, at least perturbatively in γ.

Functions α and β: perturbative expansion in γ

The polynomials Q and P corresponding to the largest eigenvalue are characterized by

$$Q(t) = t^n + \mathcal{O}(\gamma) \qquad \text{and} \qquad P(t) = 1 + \mathcal{O}(\gamma)$$

Expansion near $\gamma = 0$

$$\log\left(\frac{Q(t)}{t^n}\right) = \frac{Q_1(t)}{t^n}\gamma + \left(\frac{Q_2(t)}{t^n} - \frac{Q_1(t)^2}{2t^{2n}}\right)\gamma^2 + \dots$$

$$\log(P(t)) = P_1(t)\,\gamma + \left(P_2(t) - \frac{P_1(t)^2}{2}\right)\gamma^2 + \dots$$

Implies that

$$\alpha(t) = \log\left(\frac{q^n Q(t/q)}{Q(t)}\right) \quad \text{has only strictly negative powers in } t$$

$$\beta(t) = \log\left(\frac{P(t/q)}{P(t)}\right) \quad \text{has only strictly positive powers in } t$$

Functions α and β: Laurent expansion in t

y_i: zeros of Q (Bethe roots)
$\tilde{y}_j$: zeros of P (Bethe roots for the system with $n \leftrightarrow L-n$ and $e^{\gamma} \leftrightarrow qe^{-\gamma}$)

$$\alpha(t) = \log\left(\frac{q^n Q(t/q)}{Q(t)}\right) = \sum_{i=1}^{n}\left[\log\left(1-\frac{qy_i}{t}\right) - \log\left(1-\frac{y_i}{t}\right)\right]$$

expansion in powers of $1/t$ if $\max_i\{|y_i|, q|y_i|\} < |t|$

$$\beta(t) = \log\left(\frac{P(t/q)}{P(t)}\right) = \sum_{j=1}^{L-n}\left[\log\left(1-\frac{t}{q\tilde{y}_j}\right) - \log\left(1-\frac{t}{\tilde{y}_j}\right)\right]$$

expansion in powers of t if $|t| < \min_j\{|\tilde{y}_j|, q|\tilde{y}_j|\}$

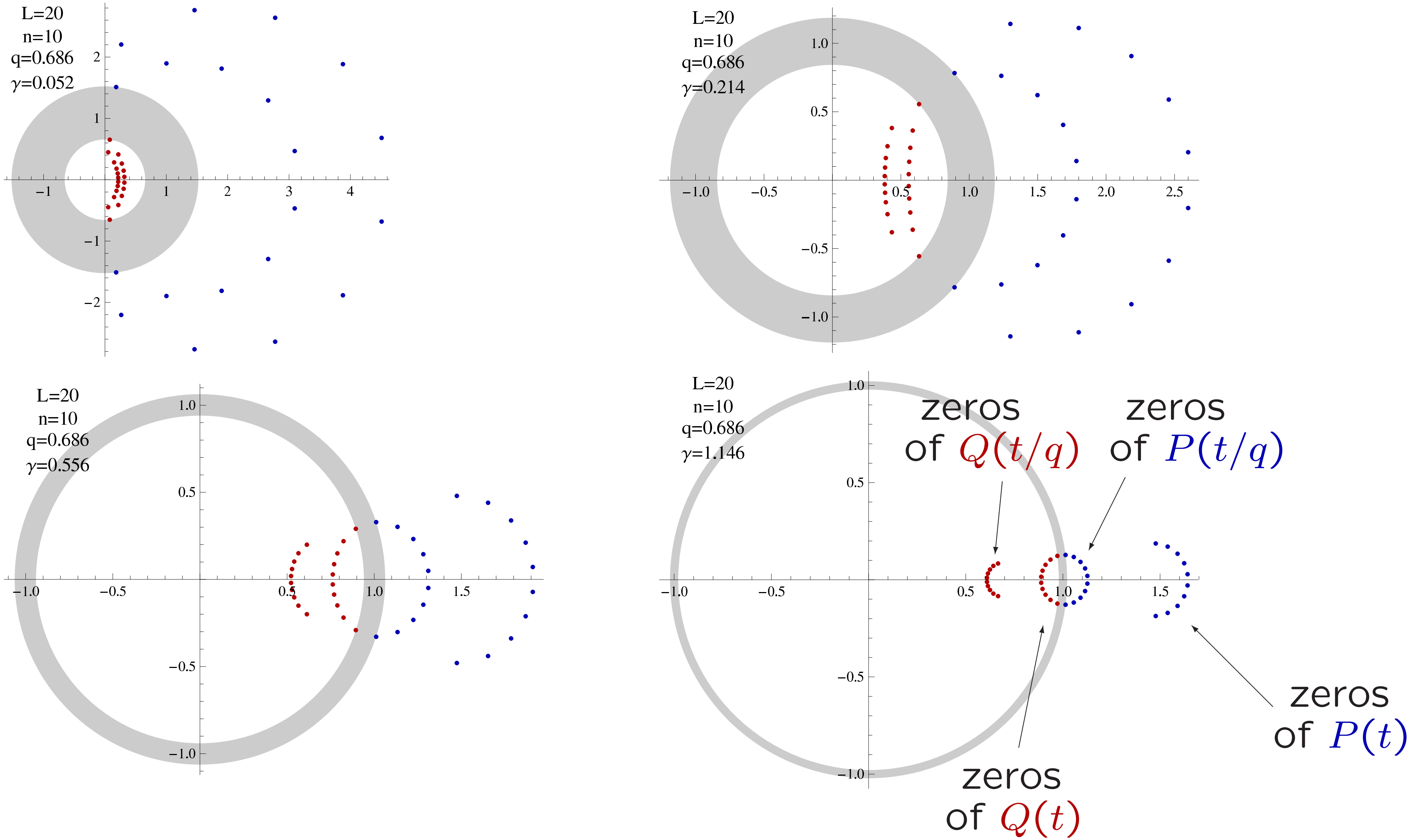
Both expansions converge in the annulus

$$\max_i\{|y_i|, q|y_i|\} < |t| < \min_j\{|\tilde{y}_j|, q|\tilde{y}_j|\}$$

if $\max_i\{|y_i|, q|y_i|\} < \min_j\{|\tilde{y}_j|, q|\tilde{y}_j|\}$, which seems to be true if $\gamma > 0$ (from a numerical solution of Baxter's equation).

Then $\alpha(t) - \beta(t)$ has a Laurent expansion with an infinity of negative and positive powers in t for t in the annulus.

Zeros of P and Q ($n = 10$, $L = 20$)



Rewriting of the quantum Wronskian

$$\alpha(t) = \log\left(\frac{q^n Q(t/q)}{Q(t)}\right) \equiv \sum_{j<0} [\alpha]_j t^j \qquad \Leftrightarrow \qquad \log\left(\frac{Q(t)}{t^n}\right) = \sum_{j<0} [\alpha]_j \frac{q^j t^j}{1-q^j}$$

$$\beta(t) = \log\left(\frac{P(t/q)}{P(t)}\right) \equiv \sum_{j>0} [\beta]_j t^j \qquad \Leftrightarrow \qquad \log(P(t)) = \sum_{j>0} [\beta]_j \frac{q^j t^j}{1-q^j}$$

$$(1 - q^n e^{-L\gamma}) Q(0) \frac{(1-t)^L}{t^n} = \frac{Q(t)}{t^n} P(t/q) - e^{-L\gamma} \frac{Q(t/q)}{(t/q)^n} P(t)$$

$$= e^{-\frac{1}{2}\left(\sum_{j<0} [\alpha]_j t^j \frac{1+q^{|j|}}{1-q^{|j|}}\right) + \frac{1}{2}\left(\sum_{j>0} [\beta]_j t^j \frac{1+q^{|j|}}{1-q^{|j|}}\right)} \left(e^{-\frac{\alpha(t)-\beta(t)}{2}} - e^{-L\gamma + \frac{\alpha(t)-\beta(t)}{2}}\right)$$

Depends on $\alpha(t)$ and $\beta(t)$ only through

$$w(t) \equiv \frac{\alpha(t)}{2} - \frac{L\gamma}{2} - \frac{\beta(t)}{2} = \log\left(\sqrt{\frac{q^n Q(t/q) P(t)}{e^{L\gamma} Q(t) P(t/q)}}\right)$$

Functional equation for w

We define the linear operator X:

$$u(t) = \sum_{j\in\mathbb{Z}} [u]_j t^j \quad \mapsto \quad X[u(t)] = \sum_{j\in\mathbb{Z}} [u]_j t^j \frac{1+q^{|j|}}{1-q^{|j|}} \qquad \left(\frac{1+q^{|0|}}{1-q^{|0|}} \equiv 1\right)$$

The functional equation for P and Q implies

$$\boxed{w(t) = \text{arcsinh}\left(C\,\frac{(1-t)^L}{t^n}\,e^{X[w(t)]}\right)}$$

where $C = -(e^{L\gamma} - q^n)\,Q(0)/2 = \mathcal{O}(\gamma)$.

$\Rightarrow$ solution order by order in C

The generating function of the cumulants of the current $E(\gamma)$ is obtained by the elimination of C between

$$E(\gamma) = -(1-q)\alpha'(1) \qquad \text{and} \qquad \gamma = \frac{\alpha(1)}{n}$$

($\alpha(t)$: negative powers in t of $w(t)$)

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Tree and forest structures

Equation for $w(t)$

$$w(t) = \text{arcsinh}\left(C \, \frac{(1-t)^L}{t^n} \, e^{X[w(t)]} \right)$$

Perturbative solution near $C = 0$

$$w(t) = \sum_{k=1}^{\infty} w_k(t) C^k$$

Expansion of $e^{X[w(t)]}$ in the equation for $w(t)$ $\Rightarrow$ tree structures

Elimination of the parameter C between

$$E(\gamma) = -(1-q)\alpha'(1) \qquad \text{and} \qquad \gamma = \frac{\alpha(1)}{n}$$

($\alpha(t)$: negative powers in t of $w(t)$)

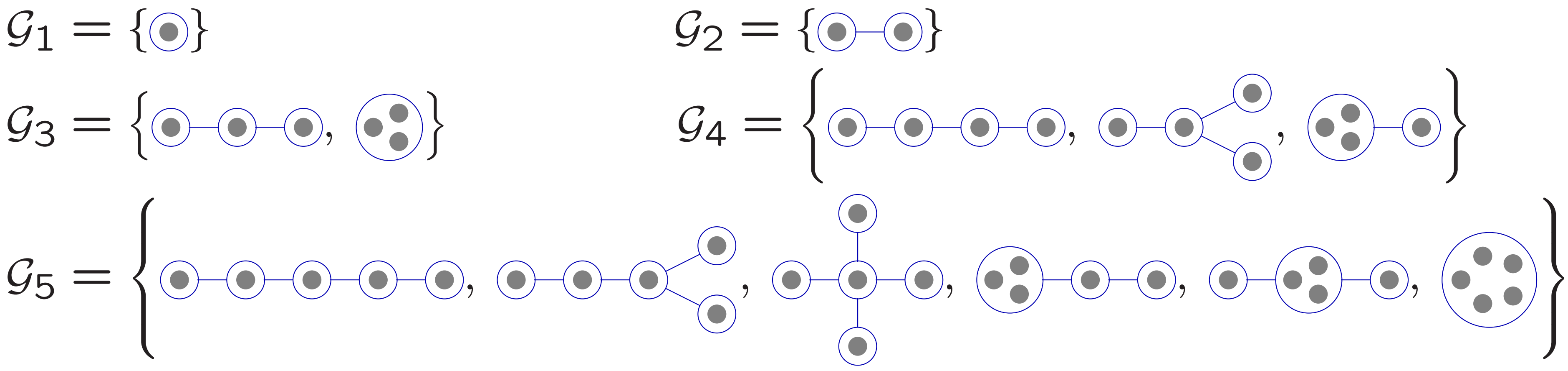
using the Lagrange inversion formula $\Rightarrow$ forest structures

Parametric expression for $E(\gamma)$

$$E(\gamma) - J\gamma = \frac{2(1-q)}{L(L-1)} \sum_{k=2}^{\infty} \left(\frac{C}{2}\right)^k \sum_{g \in \mathcal{G}_k} \frac{W_2(g)}{S(g)}$$

$$\gamma = -\frac{2}{L} \sum_{k=1}^{\infty} \left(\frac{C}{2}\right)^k \sum_{g \in \mathcal{G}_k} \frac{W_1(g)}{S(g)}$$

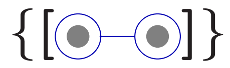
Trees with "composite nodes":



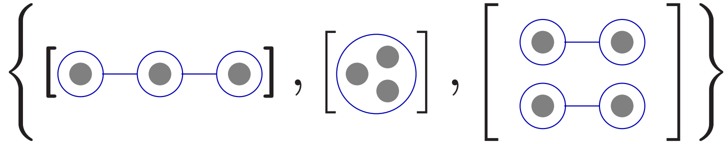
Exact formula for all the cumulants of the current

$$E_r = \frac{1-q}{L-1}\left(-\frac{L}{2}\right)^{r-1} \sum_{h \in \mathcal{H}_{r-1}} \frac{W(h)}{S(h)}$$

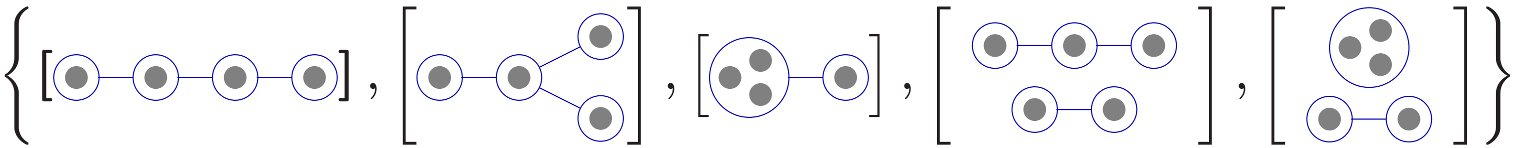
$\mathcal{H}_1 =$



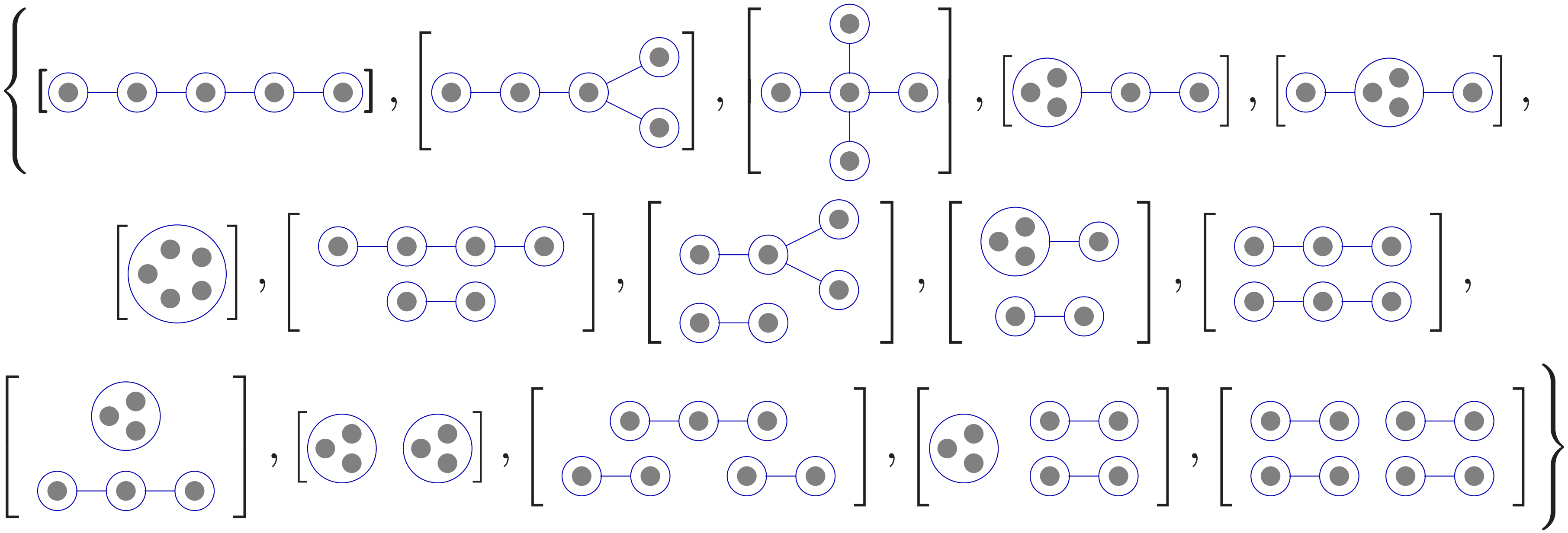
$\mathcal{H}_2 =$



$\mathcal{H}_3 =$

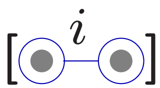


$\mathcal{H}_4 =$



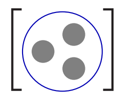
Example: first cumulants of the current

Diffusion constant:

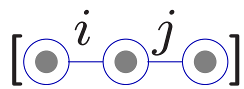


$$\frac{(L-1)D}{(1-q)L} = \sum_{i\in\mathbb{Z}} i^2 \frac{\binom{L}{n+i}\binom{L}{n-i}}{\binom{L}{n}^2} \frac{1+q^{|i|}}{1-q^{|i|}}$$

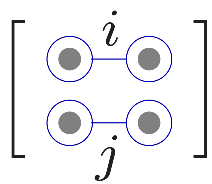
Third cumulant of the current:



$$\frac{(L-1)E_3}{(1-q)L^2} = \frac{1}{6} \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \left(i^2 + ij + j^2\right) \frac{\binom{L}{n+i}\binom{L}{n+j}\binom{L}{n-i-j}}{\binom{L}{n}^3}$$



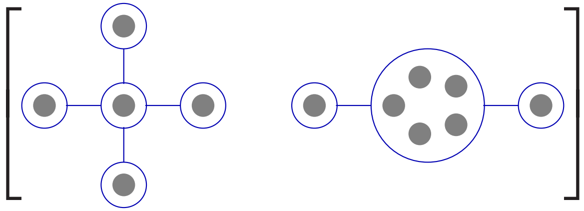
$$-\frac{3}{2} \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \left(i^2 + ij + j^2\right) \frac{\binom{L}{n+i}\binom{L}{n+j}\binom{L}{n-i-j}}{\binom{L}{n}^3} \frac{1+q^{|i|}}{1-q^{|i|}} \frac{1+q^{|j|}}{1-q^{|j|}}$$



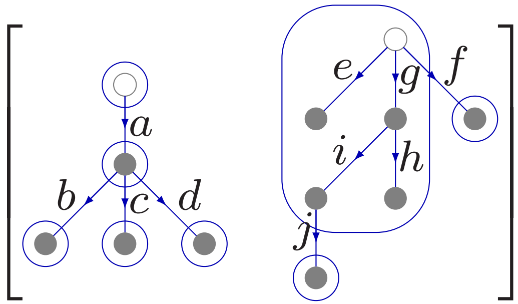
$$+\frac{3}{2} \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \left(i^2 + j^2\right) \frac{\binom{L}{n+i}\binom{L}{n-i}\binom{L}{n+j}\binom{L}{n-j}}{\binom{L}{n}^4} \frac{1+q^{|i|}}{1-q^{|i|}} \frac{1+q^{|j|}}{1-q^{|j|}}$$

Calculation of $W(h)$

$h =$



$\Rightarrow$ $\tilde{h} =$



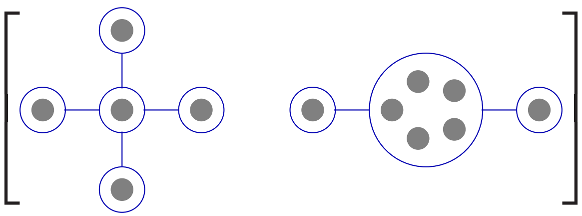
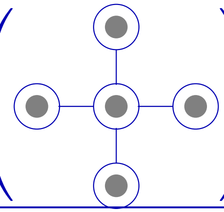
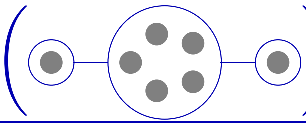
$$W(h) = \sum_{a,b,\ldots,j \in \mathbb{Z}} Q(a,b,\ldots,j) B(a,b,\ldots,j) X(a,b,\ldots,j)$$

$$\begin{aligned} Q(a,b,\ldots,j) &= (-a)^2 + (a-b-c-d)^2 + b^2 + c^2 + d^2 \\ &\quad + (-e-f-g)^2 + e^2 + f^2 + (g-h-i)^2 + h^2 + (i-j)^2 + j^2 \\ B(a,b,\ldots,j) &= \eta(-a)\eta(a-b-c-d)\eta(b)\eta(c)\eta(d) \\ &\quad \times \eta(-e-f-g)\eta(e)\eta(f)\eta(g-h-i)\eta(h)\eta(i-j)\eta(j) \\ X(a,b,\ldots,j) &= \xi(a)\xi(b)\xi(c)\xi(d) \times \xi(f)\xi(j) \end{aligned}$$

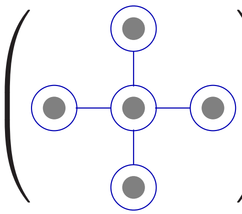
with $\eta(z) = \dfrac{\binom{L}{n+z}}{\binom{L}{n}}$ and $\xi(z) = \left| \begin{array}{ll} 1 & \text{if } z = 0 \\ \frac{1+q^{|z|}}{1-q^{|z|}} & \text{if } z \neq 0 \end{array} \right.$

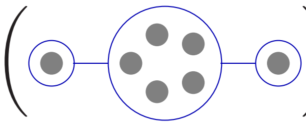
Calculation of the symmetry factors

h forest: $$S(h) = (-1)^{\text{nb trees in } h} \times \frac{\left(\begin{array}{c}\text{nb permutations}\\ \text{of the trees}\\ \text{leaving } h \text{ invariant}\end{array}\right)}{((\text{nb} \bullet \text{ in } h) - 1)!} \times \prod_{\substack{g \text{ tree}\\ \text{of } h}} \frac{S(g)}{(\text{nb} \bullet \text{ in } g)}$$

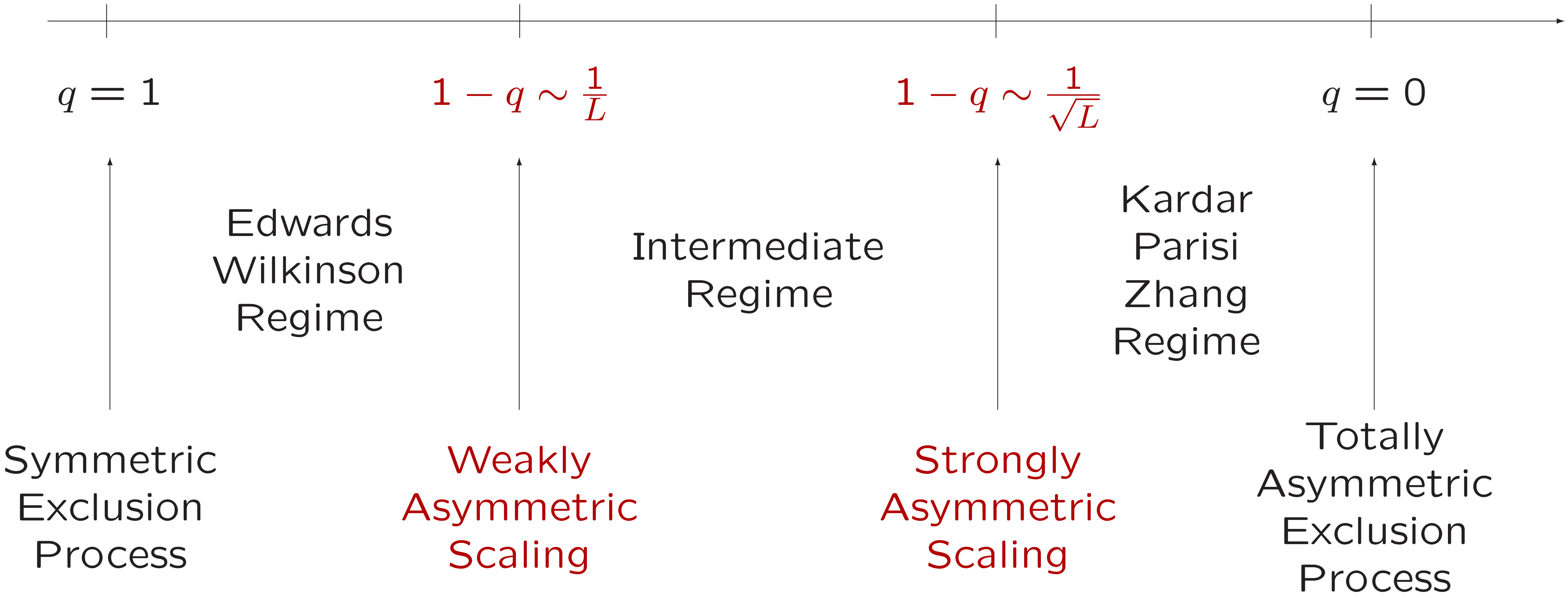
$S\Big[$  $\Big] = (-1)^2 \times \frac{1}{11!} \times \frac{S\big(\text{ }$  $\big)}{5} \times \frac{S\big(\text{ }$  $\big)}{7}$

g tree: $$S(g) = \left(\begin{array}{c}\text{nb permutations of}\\ \text{the composite nodes}\\ \text{leaving } g \text{ invariant}\end{array}\right) \times \prod_{\substack{c \text{ composite}\\ \text{node of } g}} \frac{(-1)^{\frac{|c|-1}{2}} |c|^3 |c|!}{(|c|!!)^2 |c|^{\text{nb neighbours of } c}}$$

$S\Big($  $\Big) = 4! \times 1^5$

$S\Big($  $\Big) = 2! \times 1 \times \frac{(-1)^{\frac{5-1}{2}} 5^3 5!}{(5!!)^2 5^2} \times 1$

Two interesting scalings for the asymmetry



In both weakly asymmetric and strongly asymmetric scalings, $q \to 1$ and $\Delta \to 1$ when $L \to \infty$

Weakly asymmetric scaling $1 - q \sim 1/L$

Scaling

$$1 - q \sim \frac{\nu}{L\sqrt{\rho(1-\rho)}} \qquad \text{and} \qquad \gamma \sim \frac{\mu}{\sqrt{\rho(1-\rho)L}}$$

Generating function of the cumulants of the current

$$E(\gamma) \sim \frac{\mu^2 + \mu\nu}{L} + \frac{1}{L^2}\left(-\frac{\mu^2\nu}{2\sqrt{\rho(1-\rho)}} + \varphi(\mu^2 + \mu\nu)\right) + \mathcal{O}\left(\frac{1}{L}\right)^3$$

with $\varphi[z] = \sum_{k=1}^{\infty} \frac{B_{2k-2}}{k!(k-1)!} z^k$

- B_j: Bernoulli numbers.
- Leading term (of order $1/L$) quadratic $\Rightarrow$ gaussian fluctuations.
- Sub-leading term (of order $1/L^2$): non-gaussian correction.
- $\varphi[z]$ has a non analyticity in $z = -\pi^2$.

But non-perturbative effects in γ in $E(\gamma)$. For $|\nu| > \nu_c = 2\pi$, $E(\gamma)$ becomes non-gaussian at the leading order in L: phase transition visible on the sub-leading term of $E(\gamma)$.

Strongly asymmetric scaling $1 - q \sim 1/\sqrt{L}$

Scaling

$$1 - q \sim \frac{2\Phi}{\sqrt{\rho(1-\rho)L}} \qquad \text{and} \qquad \gamma \sim \frac{\sigma}{\sqrt{\rho(1-\rho)}L^{3/2}}$$

Diffusion constant

$$D \sim 4\Phi\rho(1-\rho)L \int_0^\infty du \frac{u^2 e^{-u^2}}{\tanh(\Phi u)}$$

Third cumulant of the current

$$E_3 \sim 4\Phi\rho^{3/2}(1-\rho)^{3/2}L^{5/2} \times$$
$$\left(-\frac{\pi}{3\sqrt{3}} + 3\int_0^\infty du \int_0^\infty dv \frac{(u^2+v^2)e^{-u^2-v^2} - (u^2+uv+v^2)e^{-u^2-uv-v^2}}{\tanh(\Phi u)\tanh(\Phi v)}\right)$$

Generating function $E(\gamma)$

$$E(\gamma) \sim \frac{1}{L^2}\sum_{k=1}^\infty \frac{\sigma^k}{k!}\int_{-\infty}^\infty g_k(\Phi, \vec{u})\, du_1 \dots du_k$$

$g_k(\Phi, \vec{u})$: sum over forest structures

Conclusion

- Exact solution of Baxter's equation as a perturbative expansion in the twist parameter (eigenstate corresponding to the largest eigenvalue).
- Exact combinatorial expression for all the cumulants of the current in the asymmetric exclusion process (finite size system).

- Phase transition in the weakly asymmetric scaling: what does it mean for the six vertex model ?
- Exact solution of Baxter's equation for finite γ ? For other eigenstates ?
- Direct combinatorial calculation of the cumulants of the current (without Bethe Ansatz) ?
- Calculation of the current fluctuations for other models (open system, several species of particles) ?