Topological defects in Liouville CFT

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(overview of old work with J.-B. Zuber

and arXiv:0912:5535)

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Topological defects

(''seams'' along non contractable cycles)

algebraically defined: [VP-Zuber]

operators X in $\mathcal{H}_{\mathbf{phys}} = \oplus_{l,\bar{l}\in\mathcal{I}} Z_{l\bar{l}} \mathcal{V}_l \otimes \bar{\mathcal{V}}_{\bar{l}}$ commuting with the left and right chiral algebras of the CFT

$$[L_n, X] = [\bar{L}_n, X] = 0$$

the condition ensures that the operator X is invariant under a distorsion of the line to which it is attached.

• Classification - from the study of twisted **partition functions**, modified by the insertion of such operators. In the $\widehat{sl}(2)$ - related rational CFT - the integrable WZW model, or the minimal Virasoro - encoded in generalised ''ADE'' (Ocneanu) graphs.

• Extensions to the non-rational conformal theories like the **Liouville** CFT

- $\mathcal{H}_{\rm phys}$ - integral over continuous reps.

Problem:

• how the insertion of defects modifies the OPE of local fields

$$\Phi_{(\alpha_1,\bar{\alpha}_1)}(x_1) X \Phi_{(\alpha_2,\bar{\alpha}_2)}(x_2)_{|_{x_{12}\to 0}} \sim ? \quad \text{spectrum, OPE coeffs}$$

$\rightarrow$ study **crossing (duality) relations** for 4-point functions on the sphere with inserted defects

• in Liouville CFT the problem appears to be related - in the context of the AGT correspondence - to the search for Liouville counterparts of expectation values of generalised Wilson and 't Hooft loop operators in $\mathcal{N}=2$ supersymmetric 4d gauge theories - [AGGTV] (Alday, Gaiotto, Gukov, Tachikawa, H. Verlinde), [DGOT] (Drukker, Gomis, Okuda, Teschner) - by different approach.

Plan:

• brief review of the topological defects in the rational CFT (work with J.-B. Zuber);

• the Liouville case;

• the crossing relation for the 4-point functions modified by defects and the

Wilson-'t Hooft loop operator duality

Solutions of the commutation relations in RCFT:

linear combinations

$$X_x = \sum_{J=(j,\bar{j};\alpha,\alpha')} \frac{\Psi_x^{(j,\bar{j};\alpha,\alpha')}}{\sqrt{S_{1j}S_{1\bar{j}}}} P^{(j,\bar{j};\alpha,\alpha')}$$

of projectors, intertwiners of pairs of subspaces of $\mathcal{H}_{\text{phys}}$

$$(\mathcal{V}_j \otimes \overline{\mathcal{V}}_{\bar{j}})^{(\alpha')} \rightarrow (\mathcal{V}_j \otimes \overline{\mathcal{V}}_{\bar{j}})^{(\alpha)}, \qquad j,\bar{j} \in \mathcal{I}, \ \alpha,\alpha' = 1,2,\ldots,Z_{j\bar{j}}$$

$$P^{(j,\bar{j};\alpha,\alpha')} P^{(k,\bar{k};\beta,\beta')} = \delta_{jk}\delta_{\bar{j}\bar{k}}\delta_{\alpha'\beta} \ P^{(j,\bar{j};\alpha,\beta')}$$

Ψ_x^J - unitary matrix (complete set of defects), S_{ij} - modular matrix $\tau \to -1/\tau$,

$J = (j,\bar{j},\alpha,\alpha') \in \tilde{\mathcal{E}}$ (play the role of "exponents") $\leftrightarrow$ set of defects $\tilde{\mathcal{V}} \ni x$

same cardinality $|\tilde{\mathcal{E}}| = \sum_{l,\bar{l}} Z_{l\bar{l}}^2 = |\tilde{\mathcal{V}}|$,

Partition function on a torus in the presence of topological defects

• the multiplicity of local fields $Z_{l\bar{l}} \rightarrow \widetilde{V}_{lk;\, x_1, x_2, \ldots x_{n-1}}{}^{x_n}$ - all required to be non-negative integers (multiplicities of "defect fields"), depending on all the inserted defects,

e.g. two defects - part. function computed in two ways, analogous to the derivation of cylinder partition functions

$$Z_{y|x} = \text{tr}_{\mathcal{H}_P}(X_y^+ X_x\ \tilde{q}^{L_0 - c/24}\, \tilde{q}^{\bar{L}_0 - c/24})$$

$$= \quad \sum\nolimits_{i,k \in \mathcal{I}} \widetilde{V}_{ik^*;\, x}{}^{y}\ \chi_i(q)\, \chi_k^*(q)\,.$$

The two expressions - related by modular transformation S

$$\Rightarrow\ \widetilde{V}_{ik;\, x}{}^{y} = \sum_{j, \bar{j}, \alpha, \alpha'} \frac{S_{ij}}{S_{1j}} \frac{S_{k\bar{j}}}{S_{1\bar{j}}}\ \Psi_x^{(j,\bar{j};\alpha,\alpha')}\ \Psi_y^{(j,\bar{j};\alpha,\alpha')\,*}\,, \qquad \widetilde{V}_{ik^*;1}{}^{1} = Z_{ik}$$

$$\Rightarrow\ \widetilde{V}_{ii'}\widetilde{V}_{jj'} = \sum_{k,k'} \mathcal{N}_{ij}{}^{k}\, \mathcal{N}_{i'j'}{}^{k'}\, \widetilde{V}_{kk'}$$

$\Rightarrow$ classification of defects in RCFT - reduces to the classification of the non-negative integer valued matrix reps (nimreps) of the product of Verlinde algebras

$$\mathcal{N}_{ij}{}^{k} = \sum_{\ell \in \mathcal{I}} \frac{S_{il}}{S_{1l}} S_{jl}\, S^{*}_{kl}$$

problem - analogous to the classification of conf. boundary conditions - **nimreps** $(n_j)_a^b$

To consider many defects - exploit

Fusion algebra of defects

$$X_x\ X_y = \sum_z \widetilde{N}_{xy}{}^{z}\ X_z$$

multiplicities $\widetilde{N}_{xy}{}^{z} = \widetilde{V}_{11;xy}{}^{z}$ - the identity character contribution to the partition function with 3 defects; $\widetilde{N}_{xy}{}^{z}$ - strc consts of associative (but non-commutative algebra if some $Z_{j\bar{j}} > 1$) (Ocneanu graph algebra) ; given $\widetilde{N}$, sufficient $\widetilde{V}_{ij;1}{}^{y}$

$$\widetilde{V}_{ij;x}{}^{z} = \sum_y \widetilde{N}_{xy}{}^{z} \widetilde{V}_{ij;1}{}^{y}$$

- On a cylinder - both defects and boundaries

$$X_x|a> = \sum_c \tilde{n}_{ax}{}^c |c>$$

i.e., the defects map conformal boundary conditions to conformal boundary conditions,

$$\tilde{n}_{ax}{}^c = \sum_{l,\alpha,\beta} \psi_a^{(l,\alpha)} \frac{\Psi_x^{(l,l;\alpha,\beta)}}{\sqrt{S_{1l} S_{1\bar{l}}}} \psi_c^{(l,\beta)*}, \ \tilde{n}_x \tilde{n}_y = \sum \widetilde{N}_{xy}{}^z \, \tilde{n}_z$$

ψ - unitary matrix diagonalising Cardy multiplicity $n_{ja}{}^b = \sum_{l,\alpha} \psi_a^{(l,\alpha)} \frac{S_{jl}}{S_{1l}} \psi_b^{(l,\alpha)*}$.

- the set of multiplicities $\{\mathcal{N}_j, n_j, \widetilde{N}_x, \tilde{n}_x\}$ - determines the combinatorial data of (Ocneanu) quantum symmetry of the RCFT.

(see [Fuchs, Fröhlich, Schweigert, Runkel] for exhaustive study of topological defects)

- simplest example - a theory described by a diagonal mod invariant, i.e., with scalars only, in which all these constants coincide.

Diagonal case: $Z_{j\bar{j}} = \delta_{j\bar{j}}$

the set of defects $\tilde{\mathcal{V}}$ (and the set of boundaries) identical to the set of reps $\mathcal{I}$ of the chiral algebra, all multiplicities coincide with Verlinde fusion multiplicity $\mathcal{N}$, $\Psi = S = \psi$,

$$X_x = \sum_j \frac{S_{xj}}{S_{1j}} P^{(j,j)}$$

$$\widetilde{V}_{ij} = \mathcal{N}_i \mathcal{N}_j \,, \quad \widetilde{V}_{ij;1}{}^y = \mathcal{N}_{ij}{}^y$$

Comment:
The effect of inserting a diagonal defect operator - equivalently described by a chiral operator X_x^I ($P^{(j,j)} \to P^{(j)}$ - the operator $\sum_k |j,k><j,k|$ corresponding to the Ishibashi state)

- the twisted partition function with a defect can be interpreted alternatively in terms of chiral operators, associated with the two cycles of the torus

$$\hat{X}_x(a)\chi_j(-1/\tau) = \mathrm{Tr}_j X_x^I \tilde{q}^{L_0 - c/24} = \frac{S_{xj}}{S_{1j}}\, \chi_j(-1/\tau)$$

$$\hat{X}_x(b)\chi_k(\tau) = \sum_p \mathcal{N}_{kx}{}^p \chi_p(\tau)$$

i.e.,these chiral operators act as the Verlinde operators associated with the two cycles

• In WZW theories (integrable reps of KM algebra $\widehat{g}$) - the chiral topological defects X_x - provide the quantisation of the classical Wilson loops operator [Gaberdiel-Bachas]

$$W_x =: \text{Tr}_x e^{-\frac{i}{k}\oint_C J(z)dz} :$$

$J(z) = J^a(z)t^a$ - holomorphic current generating the affine algebra $\widehat{g}$;

identified [Alekseev, Monnier] with "generalised Casimir operators",
central elements in completion of the universal enveloping algebra of the affine KM algebra $\widehat{g}$ - constructed by [Kac] ;

eigenvalues $W_x|j\rangle = \frac{S_{xj}}{S_{1j}}|j\rangle\,,\ |j\rangle \in \mathcal{V}_j$ coincide with characters of finite dim irrep x of G

• This comment - to make contact with the chiral interpretation of other work - defects

(even diagonal) - will be treated here as 2d operators.

Liouville CFT - (non-rational) Virasoro theory with central charge $c > 25$

$$c = 1 + 6Q^2\,, \quad Q = \frac{1}{b} + b\,, \quad b - \text{real}$$

$$S = \frac{1}{4\pi}\int d^2x \sqrt{\hat{g}}(\hat{g}^{ab}\partial_a\phi\partial_b\phi + Q\phi\hat{R} \; + 4\pi\mu e^{2b\phi})$$

$V_\alpha(x) = e^{2\alpha\phi(x)}\,, \; \triangle(\alpha) = a(Q-\alpha)\,,$ continuous spectrum $\alpha \in \frac{Q}{2} + i\mathbb{R}^+$

basic strc consts known:

- 3-point bulk [DO, ZZ, T], quantum -6j -symbols (fusing matrix) [PT],
- 2-point, 3-point boundary, bulk-boundary [FZZ,ZZ, PT, H]

[actually - analogously to the diagonal rational CFT - all consts in the boundary theory are related to the basic consts of the chiral Liouville theory - add the modular matrix $S_{\alpha\beta}(p)$ of 1-point chiral correlators on the torus.]

Two types of defects [Sarkissian]

$$X_x = \int d\alpha \frac{S_{x\alpha}}{S_{0\alpha}} P^{(\alpha,\alpha)}\,, \qquad \alpha \in \frac{Q}{2} + i\mathcal{R}^+$$

"FZZ or ZZ type" modular matrices (two types of boundaries)

(up to overall coeff, not changing the eigenvalue ratios)

- continuous rep $\quad x \in \frac{Q}{2} + i\mathcal{R}^+,$

$$\widehat{S}_{x\alpha} = 2\cos\pi(2x-Q)(2\alpha-Q)$$

- degenerate Vir rep $x = x_{j,j'} = -jb - j'/b\,,\ 2j, 2j' \in \mathbf{Z}_{\geq 0}$

$$\begin{aligned} S_{x_{j,j'}\alpha} &= -4\sin\pi(2j+1)b(2\alpha-Q)\sin\frac{\pi(2j'+1)}{b}(2\alpha-Q) \\ &= \widehat{S}_{j,j'\alpha} - \widehat{S}_{-1-j,j'\alpha} \end{aligned}$$

denominator $S_{0\alpha} = S_{x_{0,0}\alpha}$

• physical correlators - different representations in different regions of the complex coordinates - correspond to different sewings of the Riemann surface by pairs of pants.

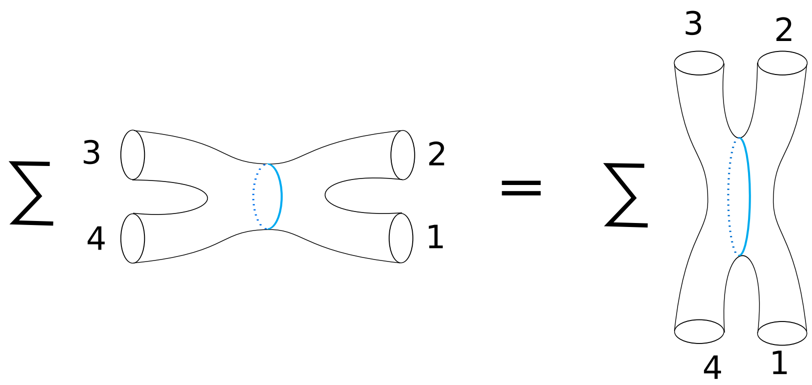
For the 4-point function - sphere with 4 punctures; s - and t- channel, valid for small moduli z (the anharmonic ratio)

$$G_4 = \int d\alpha\, C C |\mathcal{G}_\alpha(\tilde{z})|^2 = \int d\gamma\, C' C' |\mathcal{G}_\gamma(z)|^2$$

• blocks $\mathcal{G}_\alpha(\tilde{z}), \mathcal{G}_\gamma(z)$ - basis for the given pants decomposition; multivalued, under analytic continuation - transform by braiding B/ fusing F matrices.

• The full 2d correlators satisfy the **crossing symmetry** (locality)

$$\int C\, C \;\; F\, F^* = C'\, C'$$



- Correlators in the presence of defects

$$G_4 = \langle 0|\Phi_{a_4}(x_4)\,\Phi_{a_3}(x_3)\,X_x\,\Phi_{a_2}(x_2)\,\Phi_{a_1}(x_1)\,X_x^+|0\rangle$$

$$= \int d\mu(\beta)\tfrac{S_{x\beta}}{S_{0\beta}}\tfrac{S_{x0}}{S_{00}}|\mathcal{G}_\beta(a_4,a_3,a_2,a_1;\tilde{z})|^2$$

measure - accounts for a proper gauge choice

- for $\tilde{z} = \frac{z_{12}z_{34}}{z_{23}z_{14}} \to 0$ the defect diagonalizes and contributes by its eigenvalue $\langle\beta|X_x|\beta'\rangle = \frac{S_{x\beta}}{S_{0\beta}}\langle\beta|\beta'\rangle$

while in the t -channel $z = \frac{1}{\tilde{z}} \to 0$ the defect acts nontrivially in the OPE

$$G_4 = \langle 0|\Phi_{a_3}(x_3)\,X_x\,\Phi_{a_2}(x_2)\,\Phi_{a_1}(x_1)\,X_x^+\Phi_{a_4}(x_4)\,|0\rangle$$

$$= d_x \int d\mu(\gamma)d\mu(\delta)\,A^{(x)}_{\gamma,\delta}\,\mathcal{G}_\gamma(a_3,a_2,a_1,a_4;z)\mathcal{G}^*_\delta(a_3,a_2,a_1,a_4;z)$$

one has to compute the composition of the left and right braiding (fusing) transformations, modified by the defect eigenvalue

$$A^{(x)}_{\gamma,\delta} = \int d\mu(\beta) \frac{S_{x\beta}}{S_{0\beta}} F_{\beta\gamma} \begin{bmatrix} a_4 \, a_1 \\ a_3^* \, a_2 \end{bmatrix} F^*_{\beta\delta} \begin{bmatrix} a_4 \, a_1 \\ a_3^* \, a_2 \end{bmatrix} , \qquad A^{(0)}_{\gamma,\delta} = d_\gamma \, \delta(\gamma - \delta)$$

• Computed combining two basic identities in CFT:

• pentagon identity for the fusing matrix $\int FFF = FF$

• Moore-Seiberg torus identity - a relation (from the modular group of 2-point chiral correlators on the torus) involving the mod. matrix $S_{ij}(p)$ of 1-point chiral correlators $\Rightarrow$ two equations

- explicit expression for $S_{\alpha\beta}(p)$ in terms of braiding/ fusing matrix elements F

- a "Verlinde like" formula

$$\int d\beta \, \mathcal{F}_{\alpha_1\alpha_2}{}^{\beta}(p) \, \frac{S_{\beta x}}{S_{0x}} = \frac{S_{\alpha_1 x}(p)}{S_{0x}} \frac{S_{\alpha_2 x}(p^*)}{S_{0x}}$$

$\mathcal{F}_{\alpha_1\alpha_2}{}^{\beta}(p)$ - expressed in terms of F $\Rightarrow$ reproduces Verlinde fusion multiplicity for the identity operator $p = 0$

$$A^{(x)}_{\gamma,\delta} = \int d\mu(y)\, B^{(x)}_{\gamma,\delta}(y)$$

explicitly

$$B^{(x)}_{\gamma,\delta}(y) \sim F_{\alpha_3^* y^*} \begin{bmatrix} \gamma^* & \delta \\ \alpha_2 & \alpha_2 \end{bmatrix} \frac{S_{\alpha_2 x}(y^*)}{S_{0x}} \frac{e^{i\pi\triangle(y)}}{d_y} \frac{S_{\alpha_1 x}(y)}{S_{0x}} F_{\alpha_4^* y} \begin{bmatrix} \gamma & \delta^* \\ \alpha_1 & \alpha_1 \end{bmatrix}$$

• the range of y can be read from the various multiplicities involved in F and S and is dictated by the general relation

$$\widetilde{V}_{\gamma\delta^*;x}{}^{x} = \sum_y \mathcal{N}_{xy}{}^{x}\, \widetilde{V}_{\gamma\delta^*;1}{}^{y} = \sum_y \mathcal{N}_{xx^*}{}^{y}\, \mathcal{N}_{\gamma y}{}^{\delta}$$

for a degenerate defect x - the rep y is degenerate, determined by the $sl(2)\times sl(2)$ fusion rules $\mathcal{N}_{xx^*}{}^{y}$,
e.g., for $x=-jb$ the defect y takes the (integer spin) values $y=-kb\,, k=0,1,...,2j$.

Then $\mathcal{N}_{\gamma y}{}^{\delta}$ describes the fusion of a denerate with a generic representation i.e., the possible combinations (γ,δ) with $\delta=\gamma+\Gamma_y$ -shifted by the weights of the (finite) weight diagram of y.

• It follows that the spectrum of the OPE of two scalar fileds with the inserted defect X_x is described by the defect ("disorder") fields ${}^{(y)}\Phi_{(\gamma,\delta)}$;
Ex.: for $x = -b/2$ there appear 4 such fields

$$ {}^{(0)}\Phi_{(\gamma,\gamma)}\,, \quad {}^{(-b)}\Phi_{(\gamma,\gamma)}\,, \quad {}^{(-b)}\Phi_{(\gamma,\gamma-b)}\,, \quad {}^{(-b)}\Phi_{(\gamma,\gamma+b)} $$

• For a FZZ type defect the fusion multiplicities are given by integrals of densities and the spectrum of the resulting defect fields is continuous.

This explicit duality relation (and a similar computation for the 1-point scalar correlator on the torus - extending the twisted partition function) ⇒ directly related to the problem in [AGGTV], [DGOT] - on the Liouville realisation of the expectation values of the Wilson - 't Hooft loop operators in $\mathcal{N} = 2$ supersymmetric 4d theory.

• It concerns the particular degenerate defects $x = -jb$ or $x = -j'/b$, for which the defect eigenvalue takes the form of a character of finite dim irrep

$$ \frac{S_{x\alpha}}{S_{0\alpha}} = \frac{\sin(2j+1)\phi}{\sin\phi}\,, \quad \phi = \pi b(2\beta - Q) $$

According to the AGT correspondence this allows to identify the s - channel of the correlator - in which the defect diagonalises - with the expectation value of 4d generalised supersymmetric Wilson loop operator, computed on S^4 (for $b = 1$) by [Pestun] - in which the same classical character appears under the integral, times $|Z_{\text{Nekrasov}}|^2$.

Then the explicitly computed dual, t- channel of the defect correlator gives automatically the expectation value of the dual loop operator - to be identified with the expectation value of the 4d 't Hooft loop.

• thus the contribution of defect (disorder) fields ${}^y\Phi_{(\gamma,\delta)}$ describes the the expectation value of the generalised 't Hooft loop operator in the Liouville setting.

This reproduces the results of [AGGTV, DGOT] in which both loop operators were identified with chiral Verlinde operators - inserting the identity contribution of a pair of degenerate fields, then moving one of them along the curve, etc. The example of the simplest degenerate case $x = -b/2$ is worked out; [AGGTV] gives also a kind of a sketch of the duality of the two proposed Liouville correlators.

• Here - general explicit formula in terms of F and $S(y)$.

• The idea of 2d defect interpretation has been also independently proposed in the recent work [Drukker-Gaiotto-Gomis] - gives further generalisations to Toda theories. .

Main conclusion:

The notion of topological defects appears relevant also in the study of the dualities matching the 4d loop operators

The same computation - in the rational non-diagonal ADE cases:

• take the identity contribution $\gamma = \delta \to y = 0$ in the r.h.s. of the duality relation,

$\Rightarrow$ formula for the relative (to the diagonal A_{h-1} of the same Coxeter number) OPE coeffs ${d_{IJ}}^K$ of local spin operators $\Phi_{(J;\alpha)}$, $J = (j, \bar{j})$, $\alpha = 1, ..., Z_{j\bar{j}}$ [V.P.-Zuber]

$$\sum_{k,\bar{k},\gamma,\gamma'} {d_{(I^*;\alpha)(J^*;\beta)}}^{(K^*;\gamma)} \, {d_{(I;\alpha')(J;\beta')}}^{(K;\gamma')} \, \frac{\Psi_x^{(K;\gamma,\gamma')}}{\Psi_x^{(1)}} = \frac{\Psi_x^{(I;\alpha,\alpha')}}{\Psi_x^{(1)}} \, \frac{\Psi_x^{(J;\beta,\beta')}}{\Psi_x^{(1)}} \qquad (*)$$

inverting by $\Psi \to$ **sum over the set $\mathcal{V} \ni x$ of defects** - universal formula for the product of OPE coeffs; generalises to any RCFT. Interpretation - strc constants of generalised Pasquier algebra

$(*)$ generalises the (linear) formula for the scalar OPE coeffs in terms of the eigenvectors ψ_a^j of the ADE Cartan matrices - coincides with the structure constants of Pasquier algebra - introduced in the context of lattice ADE models with similar interpretation (and later rederived in the boundary CFT)

.... discussed with Claude Itzykson in my first visit to Saclay 1994 ....