DEFORMATION of KAZHDAN-LUSZTIG and MACDONALD BASES

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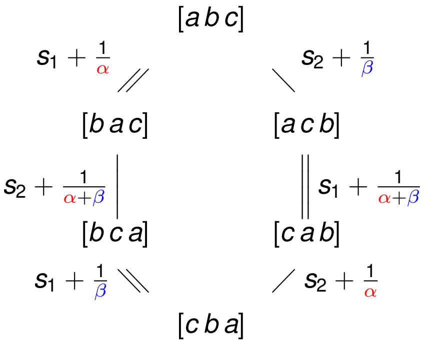
How to compute with functions of several variables?
Computers usually treat functions of $x_1, x_2, x_3, \ldots$ as functions of x_1 with coefficients in $x_2, x_3, \ldots$, and this not very illuminating to use only functions of 1 variable recursively. Fortunately, the classical groups, specially the symmetric group come to the rescousse*.
For example Alfred Young generalized the decomposition of a function of two variables into its symmetric part and antisymmetric part to any number of variables.
Essentially, his theory uses no more than the relations

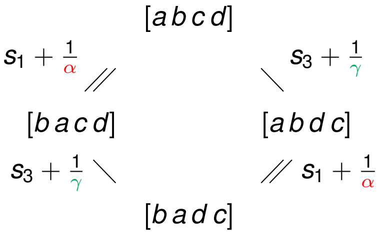
$$(1+s)(1-s) = 0 \qquad \& \qquad 1 = \frac{1+s}{2} + \frac{1-s}{2}$$

where s_1 is the simple transposition exchanging x_1, x_2,

+ the Yang-Baxter relations

Take the group algebra of $\mathfrak{S}_3$, and a spectral vector $[0, \alpha, \alpha+\beta] = [a, b, c]$; write the permutohedron, labelling the vertices with the permutations of the spectral vector, and the edges with simple transpositions + shifts





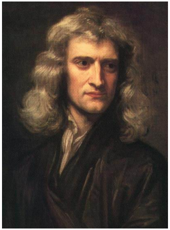
The differences of exchanged spectral values give the parameter to add to simple transpositions.

These Yang-Baxter graphs can be interpreted as describing matrices of representations satisfying

$$M_1(\alpha)M_2(\alpha+\beta)M_1(\beta) = M_2(b)M_1((\alpha+\beta)M_2(\alpha)$$

or idempotents, or bases of representations of the symmetric group, or ...

But one can have much more by replacing simple transpositions by other operators.



Isaac Newton (1643-1727)

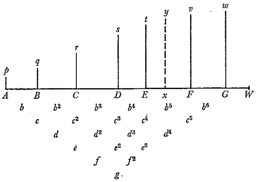
For every pair x_i, x_{i+1}, Newton defines an operator on polynomials (a divided difference) :

$$f \to f\partial_i := \frac{f(\ldots, x_i, x_{i+1}, \ldots) - f(\ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}}$$

These operators satisfy the braid relations, together with $\partial_i^2 = 0$.

vel semper decrescant. Hoc modo per bisectionem procedi potest usq; dum[30] differentiæ quartæ minores sint quam 32.[31]

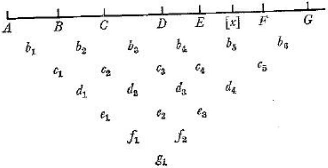
Possent aliæ hujusmodi regulæ tradi sed mallem rem omnem una regula generali complecti et ostendere quomodo series quævis in loco imperato intercalari[32] possit. Exponatur series per lineas Ap, Bq, Cr, Ds, Et, Fv, Gw &c ad lineam AG perpendiculariter erectas & intervalla terminorum per partes lineæ illius AB, BC, CD, DE &c.[33] Fac $\frac{A-B}{AB}=b$, $\frac{B-C}{BC}=b^2$, $\frac{C-D}{CD}=b^3$ &c. Item $\frac{b-b^2}{\frac{1}{2}AC}=c$, $\frac{b^2-b^3}{\frac{1}{2}BD}=c^2$, $\frac{b^3-b^4}{\frac{1}{2}CE}=c^3$ &c. Dein $\frac{c-c^2}{\frac{1}{3}AD}=d$, $\frac{c^2-c^3}{\frac{1}{3}BE}=d^2$, $\frac{c^3-c^4}{\frac{1}{3}CF}=d^3$ &[c]. Porro $\frac{d-d^2}{\frac{1}{4}AE}=e$. $\frac{d^2-d^3}{\frac{1}{4}BF}=e^2$ &c. Tunc $\frac{e-e^2}{\frac{1}{5}AF}=f$ [&c] et sic in sequentibus usq; ad ad finem operis, dividendo semper differentias primas per intervalla terminorum quorum sunt differentiæ, secundas per dimidium duorum intervallorum quibus respondent, tertias per tertiam partem trium & sic porrò pergendo usq; dum in ultimo loco differentia satis exigua sit.[34] Hoc peracto capiantur tum terminorum tum differentiarum primæ A, b, c, d, e, f, g &c. Sit differentiarum illarum numerus n,[35] locus quem intercalare oportet x, terminus intercalaris xy, et regrediendo ab ultima differentia puta g et ab ultimo terminorū ex quibus differentia illa colligebatur puta G, fac $f+\frac{g\times Gx}{n}=p$. $e+\frac{p\times Fx}{n-1}=q$. $d-\frac{q\times Ex}{n-2}=r$. $c-\frac{r\times Dx}{n-3}=s$. $b-\frac{s\times Cx}{n-4}=t$. $A-\frac{t\times Bx}{n-5}=v$,[36] pergendo semper juxta tenorem progressionis



(30) An unfinished first continuation reads 'præcedentes reg[ulæ applicari possint?]' (the preceding rules [can be applied?]).

always decrease in a regular way. In this manner a bisection procedure may be employed until(30) the fourth differences prove to be less than 32.(31)

Other rules of this kind might be presented, but I would prefer to embrace everything in one single general rule and show how any series you wish may be intercalated(32) in any place commanded. Let the series be exhibited by the lines Ap, Bq, Cr, Ds, Et, Fv, Gw, ... raised at right angles to the line AG, and the intervals of the terms by the parts AB, BC, CD, DE ... of that line.(33) Make $\frac{A-B}{AB}=b_1$, $\frac{B-C}{BC}=b_2$, $\frac{C-D}{CD}=b_3$, ...; likewise $\frac{b_1-b_2}{\frac{1}{2}AC}=c_1$, $\frac{b_2-b_3}{\frac{1}{2}BD}=c_2$,



$\frac{b_3-b_4}{\frac{1}{2}CE}=c_3$, ...; next $\frac{c_1-c_2}{\frac{1}{3}AD}=d_1$, $\frac{c_2-c_3}{\frac{1}{3}BE}=d_2$, $\frac{c_3-c_4}{\frac{1}{3}CF}=d_3$, ...; further $\frac{d_1-d_2}{\frac{1}{4}AE}=e_1$, $\frac{d_2-d_3}{\frac{1}{4}BF}=e_2$, ...; then $\frac{e_1-e_2}{\frac{1}{5}AF}=f_1$, ..., and so on in sequel till the work is finished, dividing always first differences by the intervals of the terms whose differences they are, second ones by half of the two corresponding intervals, third ones by a third of the three corresponding and so forth until the difference in the final place be slight enough.(34) When this is accomplished, take the leading quantities both of the terms and the differences, A, b_1, c_1, d_1, e_1, f_1, g_1, ..., and let those differences be n in number,(35) the place it is required to intercalate call x, the term to be intercalated xy; then, going backwards from the last difference, say g_1, and from the last of the terms, say G, from which that difference was gathered, make $f_1+g_1\times\frac{Gx}{n}=p$, $e_1+p\times\frac{Fx}{n-1}=q$, $d_1-q\times\frac{Ex}{n-2}=r$, $c_1-r\times\frac{Dx}{n-3}=s$, Since ∂_i commutes with multiplication with functions symmetrical in x_i, x_{i+1}, it is characterized by the two values

$$1\partial_i = 0 \quad \& \quad x_i\partial_i = 1 .$$

Easy to generalize to operators T_i commuting with $\mathfrak{Sym}(x_i, x_{i+1})$ and satisfying the braid relations :

$$1\,T_i = -t^{-1} \quad \& \quad x_{i+1}\,T_i = -tx_i$$

In terms of divided differences :

$$T_i = \partial_i(tx_i - t^{-1}x_{i+1}) - t^{-1}$$

The operators T_i generate the Hecke algebra and can be used to build a linear basis of the space of polynomials in $x_1, \ldots, x_n$, the basis of non-symmetric non-homogeneous Macdonald polynomials, $\{M_v : v \in \mathbb{N}^n\}$, depending on two parameters t, q. These polynomials are eigenfunctions of some operators, and can be characterized by vanishing properties.

What is the problem ?

math-ph/0703015: Quantum Knizhnik-Zamolodchikov Equation, Totally Symmetric Self-Complementary Plane Partitions and Alternating Sign Matrices Authors: P. Di Francesco, P. Zinn-Justin

cond-mat/0608160 : On polynomials interpolating between the stationary state of a O(n) model and a Q.H.E. ground state Authors: M. Kasatani, V. Pasquier

0710.5362 : Factorised solutions of Temperley-Lieb *q*KZ equations on a segment Authors: Jan de Gier, Pavel Pyatov

math-ph/0603009 : Sum rules for the ground states of the O(1) loop model on a cylinder and the XXZ spin chain Authors: P. Di Francesco, P. Zinn-Justin, J.-B. Zuber

math.QA/0507364 : Incompressible representations of the Birman-Wenzl-Murakami algebra Authors: V. Pasquier

q-alg/9508002 : Scattering matrices and affine Hecke algebras Authors: Vincent Pasquier

math-ph/0410061 : Around the Razumov-Stroganov conjecture: proof of a multi-parameter sum rule Authors: P. Di Francesco, P. Zinn-Justin

cond-mat/0101385 : The quantum symmetric *XXZ* chain at $\Delta = -\frac{1}{2}$, alternating-sign matrices and plane partitions Authors: M.T. Batchelor, J. de Gier, B. Nienhuis

In short, I would say. All the above problems involve a finite representation of the Hecke algebra $\mathcal{H}_{2n}$, corresponding to the partitions 2^n or $[n, n]$, that one can study using the operators

$$T_i(u) := T_i + \frac{t - t^{-1}}{t^{2u} - 1}$$

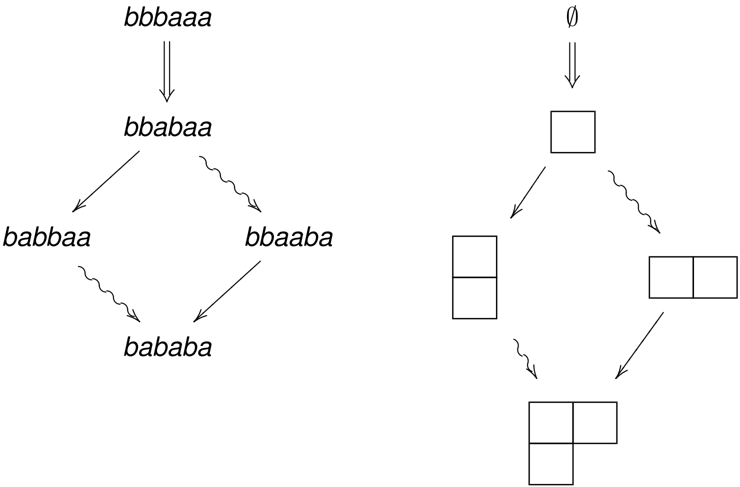
starting from the polynomial (product of t-Vandermonde)

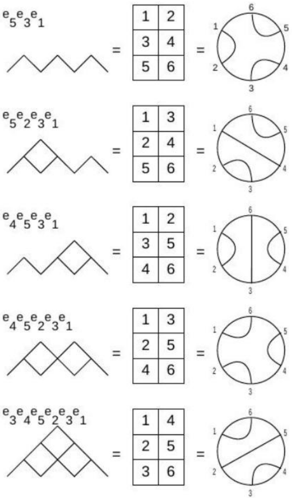
$$\prod_{1 \leq i < j \leq n} (tz_i - z_{i+1}/t) \prod_{n \leq i < j \leq 2n} (tz_i - z_{i+1}/t)$$

but also, from simply the monomial

$$z_1 \cdots z_n$$

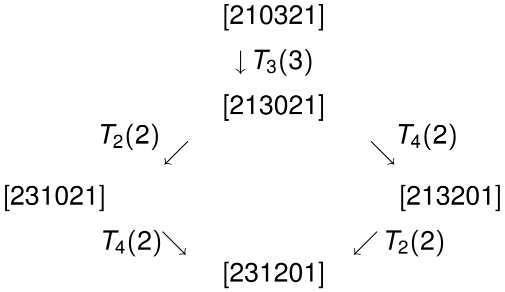
The basis of this space can be indexed in many equivalent ways :



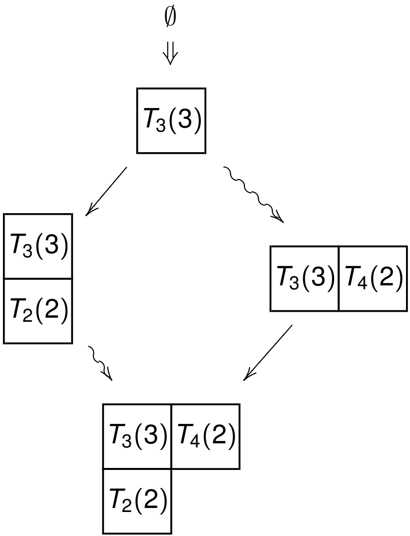


Some other ways of representing the basis

Input: a Yang-Baxter graph, an initial spectral vector, an initial polynomial.
Output: a space of polynomials with explicit basis, that one indexes by the operators creating them. One can read the action of the Hecke algebra on the space.



Yang – Baxter graph



$T_7(7)$	$T_8(6)$	$T_9(5)$	4	3	2
$T_6(6)$	$T_7(5)$	$T_8(4)$	3	2	
$T_5(5)$	$T_6(4)$	3	2		
$T_4(4)$	$T_5(3)$	2			
$T_3(3)$	2				
2					

Example of a construction using a Yang-Baxter graph : Generation of the Macdonald polynomials, corresponding to partitions contained in the staircase.

$T_7(5)$	$T_8(4)$	$T_9(2)$
$T_6(4)$	$T_7(3)$	$T_8(1)$
$T_5(3)$	$T_6(2)$	
$T_4(2)$	$T_5(1)$	
$T_3(1)$		

Kazhdan – Lusztig basis

$T_7(3)$	$T_8(2)$	$T_9(1)$
$T_3(3)$	$T_7(2)$	$T_8(1)$
$T_5(2)$	$T_6(1)$	
$T_4(2)$	$T_5(1)$	
$T_3(1)$		

Di Francesco – Zinn Justin basis

The usual basis *link pattern basis* is the Kazhdan-Lusztig basis which has many interesting properties. In particular, the sum of all the elements of the basis, specializing all the variables to 1, gives the number of ASM or TSSCPP, and there are various statistics which refine this number.

All the different bases contain as a starting point the product of t-Vandermonde :

$$\prod_{1 \le i < j \le n} (tz_i - z_{i+1}/t) \prod_{n \le i < j \le 2n} (tz_i - z_{i+1}/t)$$

which is the Macdonald polynomial of index $[n-1, \ldots, 0, n-1, \ldots, 0]$ for $q = t^6$.

Generalisation by putting different parameters in successive rows:

$T_7(7+u_6)$	$T_8(6+u_6)$	$T_9(5+u_6)$	$4+u_6$	$3+u_6$	$2+u_6$
$T_6(6+u_5)$	$T_7(5+u_5)$	$T_8(4+u_5)$	$3+u_5$	$2+u_5$	
$T_5(5+u_4)$	$T_6(4+u_4)$	$3+u_4$	$2+u_4$		
$T_4(4+u_3)$	$T_5(3+u_3)$	$2+u_3$			
$T_3(3+u_2)$	$2+u_2$				
$2+u_1$					

Integral expression

Let λ be a partition contained in the staircase, $a_i = \lambda_i + i$. Let

$$\phi_i(w) = \prod_{m=1}^{i} \frac{1}{w - z_m} \prod_{m=i+1}^{2n} \frac{1}{tw - t^{-1} z_m}$$

Then the deformed Macdonald polynomial $M_\lambda(u_1, \ldots, u_n; z_1, \ldots, z_{2n})$ is equal to

$$\Delta_t(z_1, \ldots, z_{2n}) \oint \frac{\partial w_1}{2\pi i} \ldots \oint \frac{\partial w_n}{2\pi i} \, \Delta(w_n, \ldots, w_1) \Delta_t(w_1, \ldots, w_n) \times \prod_{m=1}^{n} \frac{1}{[u_m + 1]} \left(\frac{t^{u_m+1} w_m - t^{-u_m-1} z_{a_m}}{t w_m - t^{-1} z_{a_m}} \right) \phi_{a_m}(w_m).$$

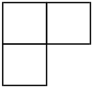
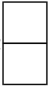
Of special interest is the *last* Macdonald polynomial, of index the staircase partition $\rho = [n{-}1, \ldots, 1]$.

Theorem.

$$M_\rho(u_1, \ldots, u_n; z_1, \ldots, z_{2n}) = \sum_{\lambda \leq \rho} c_\lambda KL_\lambda$$

sum over all the K-L basis, with explicit coefficients which are monomials of degree at most 1 in each variable $y_i = -\frac{t^{u_k} - t^{-u_k}}{t^{u_k+1} - t^{-u_k-1}}$.

For $n = 3$, for example, the expansion in terms of the K-L basis is

 $+ y_1$  $+ y_2$  $+ y_1 y_2$  $+ y_2$

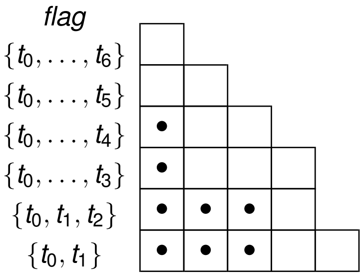
There remains the problem of specializing the polynomials in $z_1 = \cdots = z_{2n} = 1$ to obtain informations concernings TSSCPP's or ASM's. But constant terms of the type to examine are related to a fundamental scalar product on polynomials in $x_1, \ldots, x_n$:

$$(f, g) = CT\left(\prod_{i<j} 1 - x_i x_j^{-1} f(x_1, \ldots, x_n) g(x_n^{-1}, \ldots, x_1^{-1})\right)$$

compatible with divided differences, Schubert polynomials, Demazure characters, Grothendieck polynomials.
To be explicit on an example : Di Francesco and Zinn-Justin give a formula (Formula 2.7) for the number of TSSCPP according to the heights of the vertical steps.

$$N(t_0, \ldots, t_{n-1}) = CT_x\left(\prod_{i<j} \frac{(x_j - x_i)(1 + t_i x_j)}{1 - x_i x_j} \prod_i \frac{1 + t_0 x_i}{1 - x_i^2} \prod_{i=1}^{n} x_i^{-2i+2}\right)$$

One shows that $N(t_0, \ldots, t_{n-1})$ is equal to a sum of Schubert polynomials. Since Schubert polynomials can be interpreted in terms of Young tableaux, the final statement is that the constant term is equal to the sum of all staircase skew Young tableaux (inner shape made of columns of even length) statisfying a flag condition



For example, for $n = 3$

$$N(t_0, t_1, t_2)$$

$$= \begin{array}{|c|c|}\cline{1-1} t_2 & \multicolumn{1}{c}{} \\ \hline t_0 & t_0 \\ \hline \end{array} + \begin{array}{|c|c|}\cline{1-1} t_2 & \multicolumn{1}{c}{} \\ \hline t_0 & t_1 \\ \hline \end{array} + \begin{array}{|c|c|}\cline{1-1} t_2 & \multicolumn{1}{c}{} \\ \hline t_1 & t_1 \\ \hline \end{array} + \begin{array}{|c|c|}\cline{1-1} t_1 & \multicolumn{1}{c}{} \\ \hline t_0 & t_0 \\ \hline \end{array} + \begin{array}{|c|c|}\cline{1-1} t_1 & \multicolumn{1}{c}{} \\ \hline t_0 & t_1 \\ \hline \end{array} + \begin{array}{|c|c|}\cline{1-1} \bullet & \multicolumn{1}{c}{} \\ \hline \bullet & t_0 \\ \hline \end{array} + \begin{array}{|c|c|}\cline{1-1} \bullet & \multicolumn{1}{c}{} \\ \hline \bullet & t_1 \\ \hline \end{array}$$

which is, when specializing $t_2 = 1$, the enumeration of the ASM of order 3 according to the positions of top and bottom 1's, or of TSSCPP's according to the last two steps.