On one-point functions for sine-Gordon model at finite temperature.

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1. Sine-Gordon model. The Euclidean action

$$\mathcal{A}^{\rm sG} = \int \Big\{ \Big[\frac{1}{4\pi}\partial_z \boldsymbol{\varphi}(z,\bar z)\partial_{\bar z}\boldsymbol{\varphi}(z,\bar z) - \frac{\boldsymbol{\mu}^2}{\sin\pi\beta^2} e^{-i\beta\boldsymbol{\varphi}(z,\bar z)} \Big] - \frac{\boldsymbol{\mu}^2}{\sin\pi\beta^2} e^{i\beta\boldsymbol{\varphi}(z,\bar z)} \Big\} d^2z \,,$$

where $d^2z = \frac{i}{2} dz \wedge d\bar z$. In other words

$$\mathcal{A}^{\rm sG} = \mathcal{A}^{\rm CFT} + \frac{\boldsymbol{\mu}^2}{\sin\pi\beta^2} \int \Phi_{1,3}(z,\bar z) d^2 z \,.$$

CFT notation. Central charge

$$c = 1 - 6\frac{\nu^2}{1-\nu} \,, \qquad \nu = 1 - \beta^2 \,.$$

Primary fields

$$\Phi_\alpha(z,\bar z) = e^{\,\frac{\nu}{2(1-\nu)}\alpha\{i\beta\boldsymbol{\varphi}(z,\bar z)\}} \,.$$

Descendants are created by Viraroro algebras with generators $\mathbf{l}_k$, $\bar{\mathbf{l}}_k$.

There is a one-to-one correspondence between local operators in CFT and sG. The basis is

$$\mathbf{l}_{-N}\bar{\mathbf{l}}_{-\bar{N}}\Phi_\alpha(0)\,,$$

where

$$N=\{n_1,\cdots n_p\},\quad \mathbf{l}_{-N}=\mathbf{l}_{-n_1}\cdots\mathbf{l}_{-n_p}\,.$$

The soliton mass M is related to $\boldsymbol{\mu}$ by famous formula

$$\boldsymbol{\mu}\Gamma(\nu)=\left[M\frac{\sqrt{\pi}\;\Gamma(\frac{1}{2\nu})}{2\Gamma(\frac{1-\nu}{2\nu})}\right]^\nu\,.$$

2. The main problem. We want to compute the short distance asymptotics of the two-point function

$$\frac{\langle\Phi_{\alpha_1}(z,\bar z)\Phi_{\alpha_2}(0)\rangle_R^{\mathrm{sG}}}{\langle\Phi_\alpha(0)\rangle_R^{\mathrm{sG}}},\quad \alpha=\alpha_1+\alpha_2\,.$$

3. Operator product expansion.

Using the CFT perturbation theory we obtain the OPE:

$$\Phi_{\alpha_1}(z,\bar z)\Phi_{\alpha_2}(0) = \sum_{m=-\infty}^{\infty}\sum_{N,\bar N}\left(\boldsymbol{\mu}^2 r^{2\nu}\right)^{|m|} C^{m,N,\bar N}_{\alpha_1,\alpha_2}\left(\boldsymbol{\mu}^2 r^{4\nu}\right)$$
$$\times\, r^{\frac{\nu^2}{1-\nu}\alpha_1\alpha_2+2m^2(1-\nu)+2\alpha m\nu}\; z^{|N|}\bar z^{|\bar N|}\;\mathbf{l}_{-N}\bar{\mathbf{l}}_{-\bar N}\Phi_{\alpha+2m\frac{1-\nu}{\nu}}(0)\,,$$

where $|N| = \sum n_j$. The structural functions $C^{m,N,\bar N}_{\alpha_1,\alpha_2}(t)$ are power series in t^k, $k \geq 0$ with coefficients expressible through the Coulomb gas kind of integrals. So, the main problem is to compute the one-point functions:

$$\frac{\langle \mathbf{l}_{-N}\bar{\mathbf{l}}_{-\bar N}\Phi_{\alpha+2m\frac{1-\nu}{\nu}}(0)\rangle_R^{\mathrm{sG}}}{\langle \Phi_\alpha(0)\rangle_R^{\mathrm{sG}}}\,,$$

which depend on the infrared environment.

4. Fermionic basis in CFT.

Consider the space $\mathcal{V}_\alpha \otimes \overline{\mathcal{V}}_\alpha$. We have the local integrals of motion $\mathbf{i}_{2k-1}$, $\bar{\mathbf{i}}_{2k-1}$. Their action is to be factored out. We introduce the quotient spaces

$$\mathcal{V}_\alpha^{\mathrm{quo}} = \mathcal{V}_\alpha \,/\, \sum_k \mathbf{i}_{2k-1}\mathcal{V}_\alpha, \qquad \overline{\mathcal{V}}_\alpha^{\mathrm{quo}} = \overline{\mathcal{V}}_\alpha \,/\, \sum_k \bar{\mathbf{i}}_{2k-1}\overline{\mathcal{V}}_\alpha\,,$$

and consider $\mathcal{V}_\alpha^{\mathrm{quo}} \otimes \overline{\mathcal{V}}_\alpha^{\mathrm{quo}}$, which is isomorphic to the space generated by $\mathbf{l}_{-2k}$, $\bar{\mathbf{l}}_{-2k}$ acting on Φ_α.

We claim that $\mathcal{V}_\alpha^{\mathrm{quo}} \otimes \overline{\mathcal{V}}_\alpha^{\mathrm{quo}}$ can be generated from Φ_α by action of fermions

$$\boldsymbol{\beta}^*_{2k-1}, \quad \boldsymbol{\gamma}^*_{2k-1}, \quad \bar{\boldsymbol{\beta}}^*_{2k-1}, \quad \bar{\boldsymbol{\gamma}}^*_{2k-1}\,, \quad k=1,2,3,\cdots.$$

Consider the multiindex $I=\{2j_1-1,\cdots,2j_p-1\}$. We denote

$$\boldsymbol{\beta}^*_I = \boldsymbol{\beta}^*_{2j_1-1}\cdots\boldsymbol{\beta}^*_{2j_p-1}, \quad \boldsymbol{\gamma}^*_I = \boldsymbol{\gamma}^*_{2j_p-1}\cdots\boldsymbol{\gamma}^*_{2j_1-1}\,.$$

The fermionic basis is related to the usual one by the following formulae

$$\boldsymbol{\beta}^*_{I^+}\boldsymbol{\gamma}^*_{I^-}\Phi_\alpha = \prod_{2j-1\in I^+}\prod_{2k-1\in I^-} D_{2j-1}(\alpha)D_{2k-1}(2-\alpha)$$

$$\times \left[P^{\mathrm{even}}_{I^+,I^-}(\{\mathbf{l}_{-2k}\}|\Delta_\alpha,c) + d_\alpha P^{\mathrm{odd}}_{I^+,I^-}(\{\mathbf{l}_{-2k}\}|\Delta_\alpha,c)\right]\Phi_\alpha\ ,$$

where $\#(I^+)=\#(I^-)$,

$$\Delta_\alpha = \frac{\alpha(\alpha-2)\nu^2}{4(1-\nu)},$$

$$d_\alpha = \tfrac{1}{6}\sqrt{(25-c)(24\Delta_\alpha+1-c)}\,,$$

$$D_{2j-1}(\alpha) = -\sqrt{\frac{i}{\nu}}\,\Gamma(\nu)^{-\frac{2j-1}{\nu}}(1-\nu)^{\frac{2j-1}{2}}\frac{\Gamma\left(\frac{\alpha}{2}+\frac{1}{2\nu}(2j-1)\right)}{(n-1)!\Gamma\left(\frac{\alpha}{2}+\frac{1-\nu}{2\nu}(2j-1)\right)}\ .$$

The advantage of the fermionic basis is due to its consistency with the integrable perturbation.

Let us give some examples, denoting Δ_α by Δ.

$$\mathcal{P}^{\rm even}_{\{1\},\{1\}} \equiv \mathbf{l}_{-2},$$

$$2\mathcal{P}^{\rm even}_{\{1\},\{3\}} \equiv \mathbf{l}^2_{-2} + \frac{2c-32}{9}\,\mathbf{l}_{-4},$$

$$2\mathcal{P}^{\rm odd}_{\{1\},\{3\}} \equiv \frac{2}{3}\,\mathbf{l}_{-4},$$

$$3\mathcal{P}^{\rm even}_{\{1\},\{5\}} \equiv \mathbf{l}^3_{-2} + \frac{c+2-20\Delta+2c\Delta}{3(\Delta+2)}\,\mathbf{l}_{-4}\mathbf{l}_{-2}$$
$$+\,\frac{-5600\Delta+428c\Delta-6c^2\Delta+2352\Delta^2-300c\Delta^2+12c^2\Delta^2+896\Delta^3-32c\Delta^3}{60\Delta(\Delta+2)}\,\mathbf{l}_{-6}\,,$$

$$3\mathcal{P}^{\rm odd}_{\{1\},\{5\}} \equiv \frac{2\Delta}{\Delta+2}\,\mathbf{l}_{-4}\mathbf{l}_{-2} + \frac{56-52\Delta-2c+4c\Delta}{5(\Delta+2)}\,\mathbf{l}_{-6}\,,$$

$$3\mathcal{P}^{\rm even}_{\{3\},\{3\}} \equiv \mathbf{l}^3_{-2} + \frac{6+3c-76\Delta+4c\Delta}{6(\Delta+2)}\,\mathbf{l}_{-2}\mathbf{l}_{-4}$$
$$+\,\frac{-6544\Delta+498c\Delta-5c^2\Delta+2152\Delta^2-314c\Delta^2+10c^2\Delta^2-448\Delta^3+16c\Delta^3}{60\Delta(\Delta+2)}\,\mathbf{l}_{-6}\,,$$

We combine the fermions into the generating functions

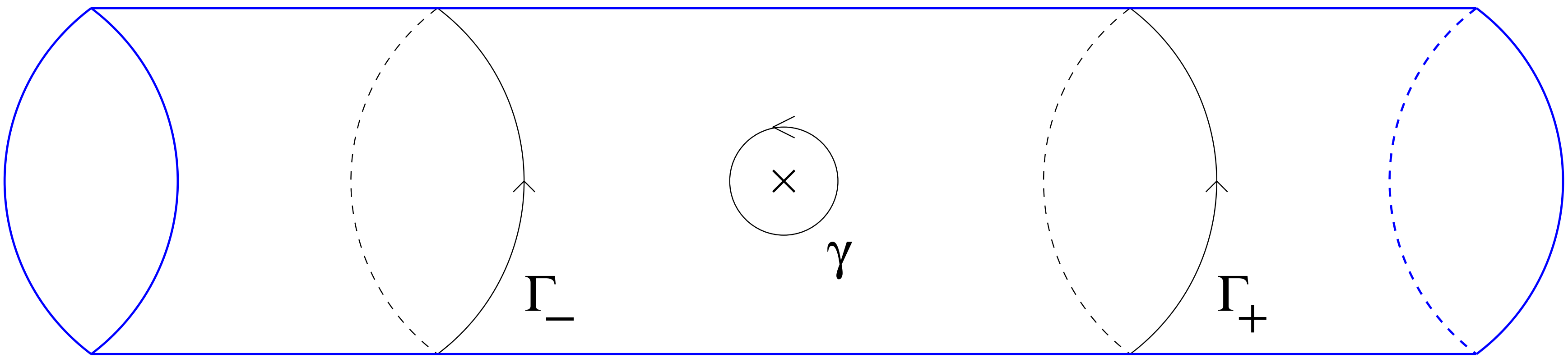
$$\boldsymbol{\beta}^*(\lambda) \simeq \sum_{j=1}^{\infty} \lambda^{-\frac{2k-1}{\nu}} \boldsymbol{\beta}^*_{2k-1}, \quad \boldsymbol{\gamma}^*(\lambda) \simeq \sum_{j=1}^{\infty} \lambda^{-\frac{2k-1}{\nu}} \boldsymbol{\gamma}^*_{2k-1} .$$

We shall see that in a weak sense these are asymptotics of analytical functions for $\lambda \to \infty$.

Similar formulae hold for the second chirality, with

$$\alpha \to 2 - \alpha , \quad \lambda \to \lambda^{-1} .$$

How do we compute. Consider the three-point function



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and define

$$\mathbf{i}_{2k-1} = \int_{\gamma} h_{2k}(z)dz, \quad I^{+}_{2k-1} = \int_{\Gamma_+} h_{2k}(z)dz, \quad I^{-}_{2k-1} = \int_{\Gamma_-} h_{2k}(z)dz\,.$$

We can compute this three-point function if the asymptotic states are eigenvectors of I^{+}_{2k-1}, I^{-}_{2k-1}. In particular, we consider the case when they are given by $\Phi_{-\kappa+1}(-\infty)$, $\Phi_{\kappa-1}(\infty)$.

Our main theorem:

$$Z_\kappa\{\boldsymbol{\beta}^*(\lambda_1)\cdots\boldsymbol{\beta}^*(\lambda_n)\boldsymbol{\gamma}^*(\mu_n)\cdots\boldsymbol{\gamma}^*(\mu_1)\Phi_\alpha(0)\} = \det\left(\omega(\lambda_i,\mu_j|\kappa,\alpha)\right)\ ,$$

where

$$\omega(\lambda,\mu|\kappa,\alpha) = \frac{1}{2\pi i}\iint dl\,dm\,\tilde{S}(l,\alpha)\tilde{S}(m,2-\alpha)\Theta(l+i0,m|\kappa,\alpha)$$
$$\times\left(\frac{e^{\frac{\pi i\nu}{2}}\Gamma(\nu)2^\nu\lambda}{(\nu\kappa)^\nu}\right)^{2il}\left(\frac{e^{\frac{\pi i\nu}{2}}\Gamma(\nu)2^\nu\mu}{(\nu\kappa)^\nu}\right)^{2im},$$

$$\tilde{S}(k,\alpha) = \frac{\Gamma\bigl(-ik+\frac{\alpha}{2}\bigr)\Gamma\bigl(\frac{1}{2}+i\nu k\bigr)}{\sqrt{2\pi}\,\Gamma\bigl(-i(1-\nu)k+\frac{\alpha}{2}\bigr)(1-\nu)^{\frac{1-\alpha}{2}}}\,.$$

We have a regular method to compute asymptotic series of $\Theta(\lambda,\mu|\kappa,\alpha)$ in κ^{-2} for $\kappa\to\infty$. We actually computed up to κ^{-8}.

Screening fermions. In weak sense $\boldsymbol{\beta}^*(\lambda)$ *etc.* are analytical functions. Introduce

$$\boldsymbol{\gamma}^*(\lambda) \simeq_{\lambda\to 0} \sum_{j=1}^{\infty} \lambda^{2j-\alpha} \boldsymbol{\gamma}^*_{\mathrm{screen},j}, \quad \bar{\boldsymbol{\beta}}^*(\lambda) \simeq_{\lambda\to\infty} \sum_{j=1}^{\infty} \lambda^{-2j+\alpha} \bar{\boldsymbol{\beta}}^*_{\mathrm{screen},j},$$

and denote by $\mathcal{H}_\alpha$ the space created from Φ_α by

$$\boldsymbol{\beta}^*_{2k-1}, \ \boldsymbol{\gamma}^*_{2k-1}, \ \bar{\boldsymbol{\beta}}^*_{2k-1}, \ \bar{\boldsymbol{\gamma}}^*_{2k-1}, \ \boldsymbol{\gamma}^*_{\mathrm{screen},j}, \ \bar{\boldsymbol{\beta}}^*_{\mathrm{screen},j}.$$

We claim that all the quotient spaces $\mathcal{V}^{\mathrm{quo}}_{\alpha+2m\frac{1-\nu}{\nu}} \otimes \overline{\mathcal{V}}^{\mathrm{quo}}_{\alpha+2m\frac{1-\nu}{\nu}}$ can be embedded into $\mathcal{H}_\alpha$.

Define the multiindices $I(m) = \{1, 2, \cdots, m\}$, $I_{\rm odd}(m) = 2I(m) - 1$, and

$$\Phi_\alpha^{(m)}(0) = i^{-m} \boldsymbol{\mu}^{2m} \prod_{j=1}^{m} \cot \tfrac{\pi\nu}{2}(2j - \alpha)\, \boldsymbol{\gamma}^*_{{\rm screen}, I(m)}\, \bar{\boldsymbol{\beta}}^*_{{\rm screen}, I(m)} \Phi_\alpha(0)\,.$$

Conjecture verified on many particular cases:

$$\begin{aligned}
&\boldsymbol{\beta}^*_{I^+} \bar{\boldsymbol{\beta}}^*_{\bar{I}^+} \bar{\boldsymbol{\gamma}}^*_{\bar{I}^-} \boldsymbol{\gamma}^*_{I^-}\, \Phi_{\alpha + 2m\frac{1-\nu}{\nu}}(0) \\
&\cong C_m(\alpha) \boldsymbol{\beta}^*_{I^+ + 2m} \boldsymbol{\gamma}^*_{I^- - 2m} \bar{\boldsymbol{\gamma}}^*_{I^- + 2m} \bar{\boldsymbol{\beta}}^*_{I^+ - 2m} \boldsymbol{\beta}^*_{I_{\rm odd}(m)} \bar{\boldsymbol{\gamma}}^*_{I_{\rm odd}(m)} \Phi_\alpha^{(m)}(0)\,,
\end{aligned}$$

where if the indices turn negative we set

$$\begin{aligned}
&\boldsymbol{\gamma}^*_{-2j+1} = \boldsymbol{\beta}_{2j-1}, \quad \bar{\boldsymbol{\beta}}^*_{-2j+1} = \bar{\boldsymbol{\gamma}}_{2j-1} \\
&[\boldsymbol{\beta}_{2j-1}, \boldsymbol{\beta}^*_{2j-1}]_+ = -i\nu \tan \frac{\pi}{2\nu}(\nu\alpha + 2j - 1), \quad \boldsymbol{\beta}_{2j-1} \Phi_\alpha^{(m)}(0) = 0\,, \\
&[\bar{\boldsymbol{\gamma}}_{2j-1}, \bar{\boldsymbol{\gamma}}^*_{2j-1}]_+ = -i\nu \tan \frac{\pi}{2\nu}(\nu\alpha - 2j + 1), \quad \bar{\boldsymbol{\gamma}}_{2j-1} \Phi_\alpha^{(m)}(0) = 0\,.
\end{aligned}$$

For the primary field we have

$$\Phi_{\alpha+2m\frac{1-\nu}{\nu}}(0) \cong C_m(\alpha) i^{-m} \boldsymbol{\mu}^{2m} \prod_{j=1}^{m} \cot \tfrac{\pi\nu}{2}(2j-\alpha)$$

$$\times\, \boldsymbol{\beta}^{*}_{I_{\rm odd}(m)} \bar{\boldsymbol{\gamma}}^{*}_{I_{\rm odd}(m)}\, \boldsymbol{\gamma}^{*}_{{\rm screen},I(m)}\, \bar{\boldsymbol{\beta}}^{*}_{{\rm screen},I(m)} \Phi_\alpha(0)\,.$$

We want to compare with CFT, namely, with Dotsenko-Fateev formula:

$$\frac{\langle \kappa-1|\Phi_{\alpha+2\frac{1-\nu}{\nu}}(0)|1+\kappa\rangle}{\langle \kappa-1|\Phi_\alpha|1+\kappa\rangle} = \boldsymbol{\mu}^2 \Gamma(\nu)^2 \cdot Y(\alpha/2+(1-\nu)/2\nu)\, W(\alpha,\kappa)\overline{W}(\alpha,\kappa)\,,$$

where

$$Y(x) = 2\nu x \cdot \frac{\Gamma^2(\nu x+1/2-\nu/2)\Gamma(\nu-2\nu x)}{\Gamma^2(1/2+\nu/2-\nu x)\Gamma(2\nu x+1-\nu)} \cdot \frac{\Gamma(-2\nu x)}{\Gamma(2\nu x)}\,,$$

$$W(\alpha,\kappa) = \frac{\Gamma(\alpha\nu/2-\nu+1+\kappa\nu)}{\Gamma(-\alpha\nu/2+\nu+\kappa\nu)}\,, \quad \overline{W}(\alpha,\kappa) = W(\alpha,-\kappa)\,.$$

First thing we have to check is

$$(\kappa\nu)^{-\alpha\nu m-(1-\nu)m^2+\nu m}\prod_{j=0}^{m-1}W(\alpha+2j\tfrac{1-\nu}{\nu},\kappa)$$

$$=\frac{\det\Theta(\frac{i}{2\nu}(2p-1),-i(q-\frac{\alpha}{2})|\ \kappa\ ,\alpha)|_{p,q=1,\cdots m}}{\det\Theta(\frac{i}{2\nu}(2p-1),-i(q-\frac{\alpha}{2})|\infty,\alpha)|_{p,q=1,\cdots m}}\,.$$

We check this for $m=1,2$ up to κ^{-8}.
Then we find the normalisation

$$C_m(\alpha)=\prod_{j=1}^{m}F(\tfrac{\alpha}{2}+j\tfrac{1-\nu}{\nu})\,,$$

where

$$F(x)=\nu\Gamma(\nu)^{4x}\ \frac{\Gamma(-2\nu x)}{\Gamma(2\nu x)}\frac{\Gamma(x)}{\Gamma(x+1/2)}\frac{\Gamma(-x+1/2)}{\Gamma(-x)}i\cot\pi x\,.$$

To summarise we conclude that the basis of $\mathcal{V}^{\mathrm{quo}}_{\alpha+2m\frac{1-\nu}{\nu}}\otimes\overline{\mathcal{V}}^{\mathrm{quo}}_{\alpha+2m\frac{1-\nu}{\nu}}$ embedded into $\mathcal{H}_\alpha$ the basis being

$$\boldsymbol{\beta}^*_{I^+}\bar{\boldsymbol{\beta}}^*_{\bar{I}^+}\bar{\boldsymbol{\gamma}}^*_{\bar{I}^-}\boldsymbol{\gamma}^*_{I^-}\,\Phi^{(m)}_\alpha(0)\,.$$

with $\#(I^+)=\#(I^-)+m$, $\#(\bar{I}^-)=\#(\bar{I}^+)+m$.

5. One-point functions for sine Gordon.

Let us rewrite the OPE (modulo action of local integrals) in the fermionic basis:

$$\Phi_{\alpha_1}(z,\bar{z})\Phi_{\alpha_2}(0)=\sum_{m=-\infty}^{\infty}\left(\boldsymbol{\mu}^2r^{2\nu}\right)^m r^{2m^2(1-\nu)+2\alpha m\nu}$$

$$\times\sum_{\substack{\#(I^+)=\#(I^-)+m\\ \#(\bar{I}^-)=\#(\bar{I}^+)+m}}\widetilde{C}^{I^+,I^-,\bar{I}^+,\bar{I}^-}_{\alpha_1,\alpha_2}\left(\boldsymbol{\mu}^4r^{4\nu}\right)z^{|I^+|+|I^-|}\bar{z}^{|\bar{I}^+|+|\bar{I}^-|}\boldsymbol{\beta}^*_{I^+}\,\bar{\boldsymbol{\beta}}^*_{\bar{I}^+}\,\bar{\boldsymbol{\gamma}}^*_{\bar{I}^-}\,\boldsymbol{\gamma}^*_{I^-}\,\Phi^{(m)}_\alpha(0)\,.$$

TBA data. Destri-DeVega equation:

$$\frac{1}{i}\log\mathfrak{a}(\zeta)=\pi MR(\zeta^{1/\nu}-\zeta^{-1/\nu})-2\mathrm{Im}\int\limits_0^\infty R(\zeta/\xi)\log(1+\mathfrak{a}(\xi e^{+i0}))\frac{d\xi^2}{\xi^2}\,,$$

where as usual it is convenient to define $R(\zeta)$ through more general object:

$$R(\zeta,\alpha)=\int\limits_{-\infty}^{\infty}\zeta^{2ik}\widehat{R}(k,\alpha)\frac{dk}{2\pi},$$

$$\widehat{R}(k,\alpha)=\frac{\sinh\pi((2\nu-1)k-i\alpha/2)}{2\sinh\pi((1-\nu)k+i\alpha/2)\cosh(\pi\nu k)}\,,$$

Then

$$R(\zeta)=R(\zeta,0)\,.$$

Introduce the measure

$$dm(\zeta) = 2\mathrm{Re}\left(\frac{1}{1+\mathfrak{a}(\zeta e^{-i0})}\right)\frac{d\zeta^2}{\zeta^2}\,,$$

and its moment

$$G(k) = \int_0^\infty \zeta^{-2ik} dm(\zeta)\,.$$

The measure $dm(\zeta)$ decreases exponentially for $\zeta \to 0$ and $\zeta \to \infty$, hence G(k) is an entire function.

Define $\Theta_R^{\mathrm{sG}}(l, m|\alpha)$ as solution to the integral equation

$$\Theta_R^{\mathrm{sG}}(l, m|\alpha) + G(l+m) + \int\limits_{-\infty}^{\infty} G(l-k)\widehat{R}(k,\alpha)\Theta_R^{\mathrm{sG}}(k, m|\alpha)\frac{dk}{2\pi} = 0\,.$$

Main formula.

$$\frac{\langle \boldsymbol{\beta}^*_{I^+}\bar{\boldsymbol{\beta}}^*_{\bar{I}^+}\bar{\boldsymbol{\gamma}}^*_{\bar{I}^-}\boldsymbol{\gamma}^*_{I^-}\Phi^{(m)}_\alpha(0)\rangle^{\mathrm{sG}}_R}{\langle\Phi_\alpha(0)\rangle^{\mathrm{sG}}_R}$$
$$= \boldsymbol{\mu}^{2m\alpha-2m^2+\frac{1}{\nu}\left(|I^+|+|I^-|+|\bar{I}^+|+|\bar{I}^-|\right)}\,\mathcal{D}^{\mathrm{sG}}_R\left(I^+\cup(-\bar{I}^+)\mid I^-\cup(-\bar{I}^-)\right)\ ,$$

with the requirements $\#(I^+)=\#(I^-)+m$, $\#(\bar{I}^+)+m=\#(\bar{I}^-)$.

We define for $\#(A)=\#(B)=n$

$$\mathcal{D}^{\mathrm{sG}}_R(A|B)=\prod_{j=1}^{n}\mathrm{sgn}(a_j)\mathrm{sgn}(b_j)\left(\frac{i}{2\pi\nu^2}\right)^n$$
$$\times\det\left(\Theta^{\mathrm{sG}}_R\left(\tfrac{ia_j}{2\nu},\tfrac{ib_k}{2\nu}|\alpha\right)-\mathrm{sgn}(a_j)\delta_{a_j,-b_k}2\pi\nu\cot\tfrac{\pi}{2\nu}(a_j+\nu\alpha)\right)\Big|_{j,k=1,\cdots,n}\ .$$

6. Checks.

Consistency relations. There are many relations following from different possibilities to compute things. Recall

$$\boldsymbol{\beta}^*_{I^+}\bar{\boldsymbol{\beta}}^*_{\bar{I}^+}\bar{\boldsymbol{\gamma}}^*_{\bar{I}^-}\boldsymbol{\gamma}^*_{I^-}\Phi_{\alpha+2m\frac{1-\nu}{\nu}}(0)$$
$$\cong C_m(\alpha)\boldsymbol{\beta}^*_{I^++2m}\boldsymbol{\gamma}^*_{I^--2m}\bar{\boldsymbol{\gamma}}^*_{I^-+2m}\bar{\boldsymbol{\beta}}^*_{I^+-2m}\boldsymbol{\beta}^*_{I_{\mathrm{odd}}(m)}\bar{\boldsymbol{\gamma}}^*_{I_{\mathrm{odd}}(m)}\Phi^{(m)}_\alpha(0)\,.$$

All the consistency relations follow from

$$\Theta^{\mathrm{s}G}_R(l,m|\alpha+2\tfrac{1-\nu}{\nu})$$
$$=\Theta^{\mathrm{s}G}_R(l+\tfrac{i}{\nu},m-\tfrac{i}{\nu}|\alpha)-\frac{\Theta^{\mathrm{s}G}_R(l+\frac{i}{\nu},-\frac{i}{2\nu}|\alpha)\Theta^{\mathrm{s}G}_R(\frac{i}{2\nu},m-\frac{i}{\nu}|\alpha)}{\Theta^{\mathrm{s}G}_R(\frac{i}{2\nu},-\frac{i}{2\nu}|\alpha)-2\pi\nu\cot\frac{\pi}{2}(\alpha+\frac{1}{\nu})}\,,$$

which we prove using the integral equation.

Lukyanov-Zamolodchikov formula. Notice that

$$\Theta_{\infty}^{\mathrm{s}G}(l,m|\alpha)=0\,.$$

So, we have

$$\frac{\langle\Phi_{\alpha+2m\frac{1-\nu}{\nu}}(0)\rangle_{\infty}^{\mathrm{sG}}}{\langle\Phi_{\alpha}(0)\rangle_{\infty}^{\mathrm{sG}}}=\boldsymbol{\mu}^{2\alpha m+\frac{1-\nu}{\nu}m^2}C_m(\alpha)\left(i\nu^{-1}\right)^m\prod_{j=1}^{m}\cot\tfrac{\pi}{2\nu}(2j-1+\nu\alpha)\,.$$

Using explicit formula fo $C_m(\alpha)$ we find perfect agreement with Lukyanov-Zamolodchikov formula:

$$\langle\Phi_{\alpha}(0)\rangle_{\infty}^{\mathrm{sG}}=\left[M\frac{\sqrt{\pi}\;\Gamma(\frac{1}{2\nu})}{2\Gamma(\frac{1-\nu}{2\nu})}\right]^{\frac{\nu^2\alpha^2}{2(1-\nu)}}$$

$$\times\exp\Bigl(\int_0^{\infty}\Bigl(\frac{\sinh^2(\nu\alpha t)}{2\sinh(1-\nu)t\sinh t\cosh\nu t}-\frac{\nu^2\alpha^2}{2(1-\nu)}e^{-2t}\Bigr)\frac{dt}{t}\Bigr)\,.$$

Fateev-Fradkin-Lukyanov-Zamolodchikov-Zamolodchikov formula. Again $R=\infty$. For the simplest non-trivial for $R=\infty$ descendent we have

$$\boldsymbol{\beta}^*_1\boldsymbol{\gamma}^*_1\bar{\boldsymbol{\beta}}^*_1\bar{\boldsymbol{\gamma}}^*_1\Phi_\alpha(0)=\left(D(\alpha)D(2-\alpha)\right)^2\mathbf{l}_{-2}\bar{\mathbf{l}}_{-2}\Phi_\alpha(0)\,.$$

Using our main formula and the formula for $D_1(\alpha)$ we obtain FFLZZ formula:

$$\frac{\langle\mathbf{l}_{-2}\bar{\mathbf{l}}_{-2}\Phi_\alpha(0)\rangle^{\rm sG}_\infty}{\langle\Phi_\alpha(0)\rangle^{\rm sG}_\infty}=-\left[M\frac{\sqrt{\pi}\,\Gamma(\frac{1}{2\nu})}{2\sqrt{1-\nu}\,\Gamma(\frac{1-\nu}{2\nu})}\right]^4$$
$$\times\frac{\Gamma(-\frac12+\frac\alpha2+\frac1{2\nu})\Gamma(\frac12-\frac\alpha2+\frac1{2\nu})\Gamma(1-\frac\alpha2-\frac1{2\nu})\Gamma(\frac\alpha2-\frac1{2\nu})}{\Gamma(\frac32-\frac\alpha2-\frac1{2\nu})\Gamma(\frac12+\frac\alpha2-\frac1{2\nu})\Gamma(\frac\alpha2+\frac1{2\nu})\Gamma(1-\frac\alpha2+\frac1{2\nu})}\,.$$

Zamolodchikov formula. A. Zamolodchikov proves that for any two-dimensional Eucledian QFT on a cylinder the following formula holds:

$$\langle T_{z,z}T_{\bar{z},\bar{z}}\rangle \;=\; \langle T_{z,z}\rangle\langle T_{\bar{z},\bar{z}}\rangle \;-\; \langle T_{z,\bar{z}}\rangle^2\,.$$

This formula follows from

$$\langle \mathbf{l}_{-2}\bar{\mathbf{l}}_{-2}\cdot 1\rangle_R^{\mathrm{sG}} = \frac{M^4}{64\nu^2}\begin{vmatrix} \Theta_R^{\mathrm{sG}}(\frac{i}{2\nu},\frac{i}{2\nu}|0) & 2\pi\nu\cot\frac{\pi}{2\nu}+\Theta_R^{\mathrm{sG}}(\frac{i}{2\nu},-\frac{i}{2\nu}|0) \\ 2\pi\nu\cot\frac{\pi}{2\nu}+\Theta_R^{\mathrm{sG}}(-\frac{i}{2\nu},\frac{i}{2\nu}|0) & \Theta_R^{\mathrm{sG}}(-\frac{i}{2\nu},-\frac{i}{2\nu}|0)\end{vmatrix},$$

$$\langle \mathbf{l}_{-2}\cdot 1\rangle_R^{\mathrm{sG}} = \frac{M^2}{8\nu}\Theta_R^{\mathrm{sG}}(\tfrac{i}{2\nu},\tfrac{i}{2\nu}|0)\,,\quad \langle \bar{\mathbf{l}}_{-2}\cdot 1\rangle_R^{\mathrm{sG}} = \frac{M^2}{8\nu}\Theta_R^{\mathrm{sG}}(-\tfrac{i}{2\nu},-\tfrac{i}{2\nu}|0)\,,$$

$$2\pi\nu\frac{\boldsymbol{\mu}^2}{\sin\pi\nu}\langle\Phi_{2\frac{1-\nu}{\nu}}\rangle_R^{\mathrm{sG}} = \frac{M^2}{8\nu}\left(2\pi\nu\cot\tfrac{\pi}{2\nu}+\Theta_R^{\mathrm{sG}}(\tfrac{i}{2\nu},-\tfrac{i}{2\nu}|0)\right)\,.$$