Positivity proofs and integrable models

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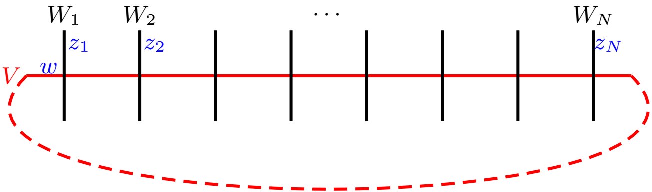
1. Generalized Heisenberg spin chains
2. New combinatorics and the completeness problem
3. New combinatorics and and the eigenvalue problem



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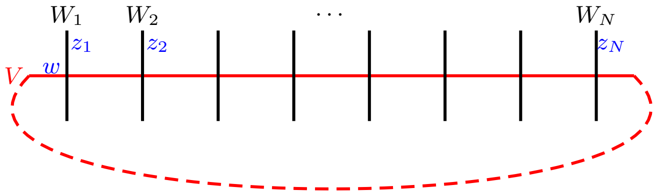
Generalized Inhomogeneous Heisenberg Spin chain



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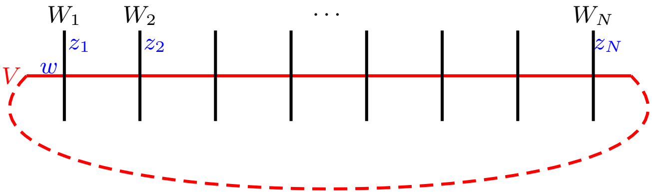

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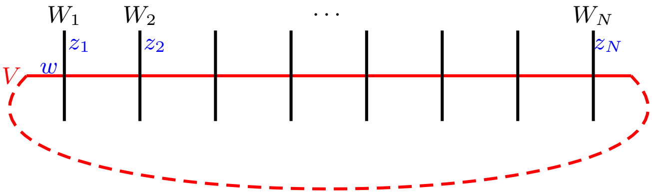

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- Define a transfer matrix $T_V(w) = \mathrm{Trace}_V \overleftarrow{\prod} R_{W_i,V}$.
- YBE $\Longrightarrow [T_V(w), T_{V'}(w')] = 0$ for any choice of representations. $\Longrightarrow$ The inhomogeneous, generalized Heisenberg spin chain is integrable.

Fact: The Bethe ansatz "works well" when V, W_i are special (KR-modules) [Kulish-Reshetikhin, Kirillov,...].

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The recursion relations are discrete integrable systems, solvable using an auxiliary statistical model.

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- $\mathbf{m}$ are non-negative integers $\{m_{i,k}\}$ with $1 \leq i \leq r$.
- The sum is restricted by "zero weight condition" and positivity of vacancy numbers.

Completeness theorem

Theorem (Hatayama et al 1999 + Di-Francesco-K. 2007)

If the characters of W_i satisfy the Q-system recursion relation, then

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The Q-system recursion relation for A_r is

$$Q_{i,k+1}Q_{i,k-1} = Q_{i,k}^2 - Q_{i+1,k}Q_{i-1,k}, \qquad 1 \leq i \leq r, \quad k \geq 1,$$

where

- $Q_{0,k} = Q_{r+1,k} = 1$ by convention;
- Boundary conditions: $Q_{i,0} = 1$ and $Q_{i,1} = \mathrm{char} V(\omega_i) =$ characters of the fundamental representations.

Q-system as an integrable discrete dynamical system

Drop the boundary condition $Q_{i,0} = 1$ and renormalize $x_{i,k} = (-1)^{\lfloor i/2 \rfloor} Q_{i,k}$:

$$x_{i,k+1} x_{i,k-1} = x_{i,k}^2 + x_{i+1,k} x_{i-1,k}, \quad x_{0,k} = x_{r+1,k} = 1, \quad k \in \mathbb{Z}, 1 \leq i \leq r$$

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- Discrete dynamical system for r functions x_i of the discrete time parameter k.

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Theorem (K.07)

For any Cartan matrix C of a simple Lie algebra $\mathfrak{g}$, the associated Q-system equations are mutations in a cluster algebra with trivial coefficients, and exchange matrix $B = \begin{pmatrix} C^t - C & -C^t \\ C & 0 \end{pmatrix}$.

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Theorem (Di-Francesco,K.)

The system is integrable, solvable, solutions are partition functions of paths on a weighted graph.

What are cluster algebras?

A rank r cluster algebra [Fomin, Zelevinsky 2000] is an algebra generated by commutative variables:

- "Clusters" of r variables $(x_1(t), ..., x_r(t))$ and an exchange matrix B live on each node t of a regular r-tree.

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- If an edge labeled i connects node t with node t' then the clusters are related by a rational expression:

$$x_i(t')x_i(t) = \prod_j x_j(t)^{[B_{ji}]_+} + \prod_j x_j(t)^{[-B_{ji}]_+}, \qquad x_{j\neq i}(t') = x_j(t).$$

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Conjecture

These polynomials have positive coefficients.

The Q-system is an integrable sub-cluster algebra

Our system has more structure than a cluster algebra: It is **integrable**

- The system has r integrals of the motion (functions of $x_{i,k}$ which are independent of k).

  Example: For A_1, $C_k = C = x_{1,k-1}x_{1,k}^{-1} + x_{1,k}x_{1,k-1}^{-1} + x_{1,k}^{-1}x_{1,k-1}^{-1}$ is independent of k.

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- The Q-system is **solvable:** $x_{1,k}$ satisfied a linear recursion relation with constant coefficients.

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  Solutions $x_{1,k}$ are partition functions of weighted paths on a graph; for A_r with $r > 1$, $x_{i,k}$ are P.F. of families of i non-intersecting paths on this graph.

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- The weights are positive so this proves positivity of the solutions (conjectured for cluster algebra).

Example: The solution for the A_1 Q-system as path PF

For A_1,

$$x_{1,k+1}x_{1,k-1} = x_{1,k}^2 + 1.$$

Solution to linear recursion relation is

$$\sum_{k\geq 0} x_{1,k}t^k = \frac{x_{1,0}}{1 - t\frac{y_1}{1-t\frac{y_2}{1-ty_3}}}$$

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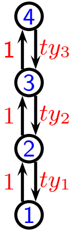
The generating function on weighted paths from node 1 to itself on the graph:



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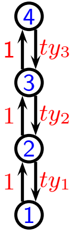

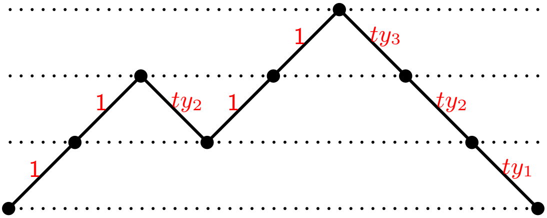

$$wt(P) = t^4 y_1 y_2^2 y_3$$

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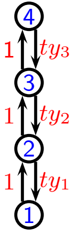
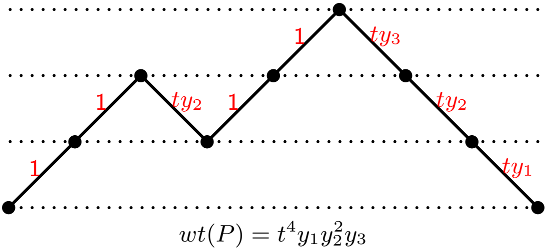
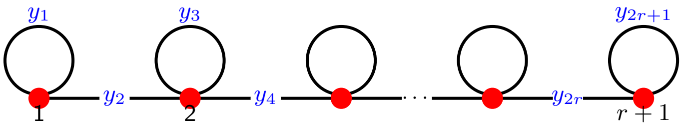




$G_r =$ 

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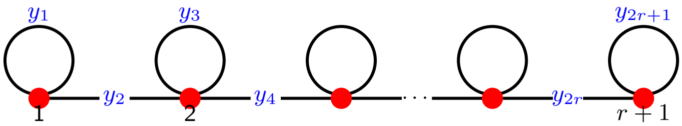

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- Nontrivial weights going from right to left:

$$y_i = y_{i,0} = \begin{cases} \frac{x_{i/2+1,0}\,x_{i/2-1,1}}{x_{i/2,0}\,x_{i/2,1}} & i \text{ even;} \\ \frac{x_{(i+1)/2,1}\,x_{(i-1)/2,0}}{x_{(i+1)/2,0}\,x_{(i-1)/2,1}} & i \text{ odd,} \end{cases}$$

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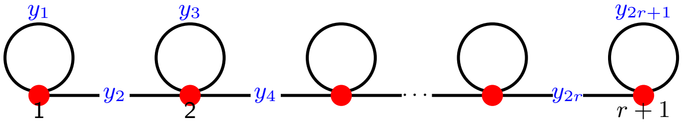
$G_r =$ 

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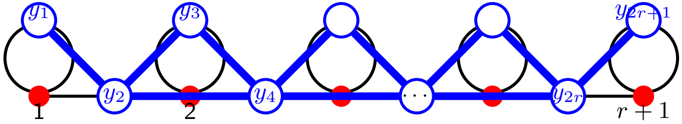
$G_r =$ 

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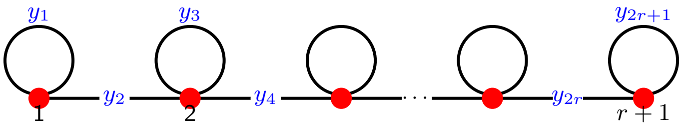
$G_r =$ 

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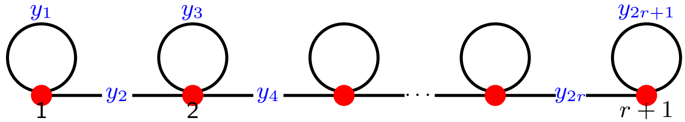

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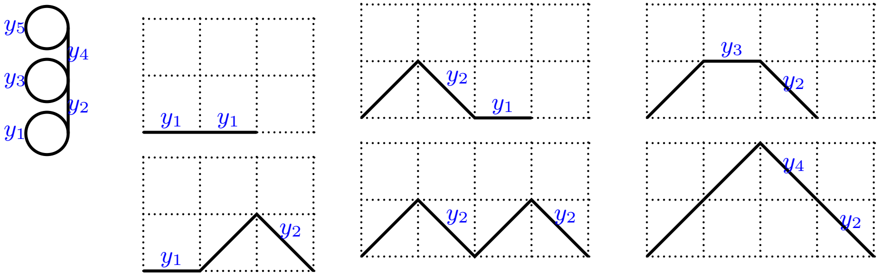
Example

For A_2 we have the following paths contributing to $x_{1,3}$ on the graph G_2



$$\frac{x_{1,3}}{x_{1,0}} = (1 + y_1 Z_{1,1})[3] = y_1 Z_{1,1}[2] = y_1(y_1^2 + 2y_1y_2 + y_2^2 + y_3y_2 + y_4y_2).$$

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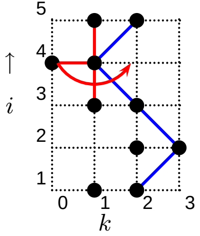

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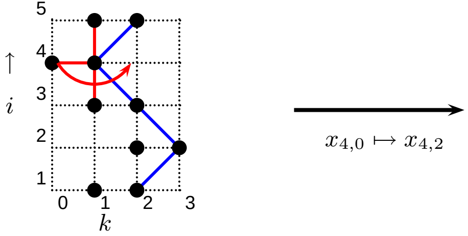

$x_{4,0} \mapsto x_{4,2}$

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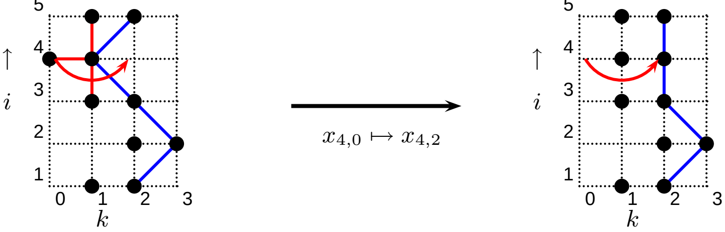

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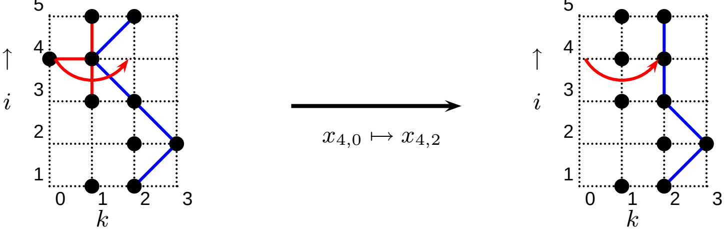

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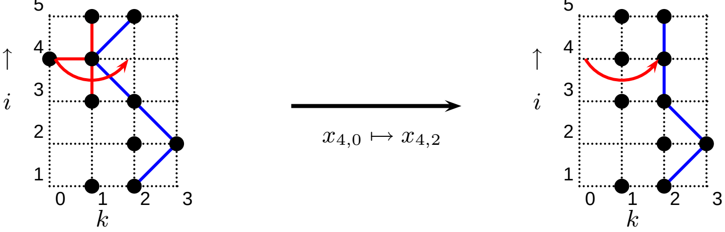

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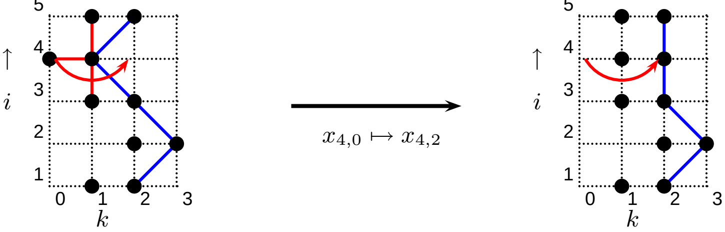

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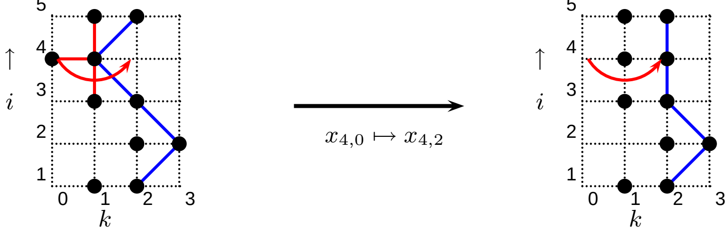

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Variables $x_{1,k}$ as a function of $\mathbf{x_m}$

Definition (The weights $z_i(\mathbf{x_m})$)

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Proof of positivity of $x_{i,k}$ follows from LGV.

The T-system for A_r

$$T_{i,j,k+1}T_{i,j,k-1} = T_{i,j+1,k}T_{i,j-1,k} - T_{i+1,j,k}T_{i-1,j,k}$$

- Satisfied by the transfer matrices $T_{i,j,k} = T_V$: auxiliary space $V = V_{i\omega_k}(j)$ ($j \sim$ spectral parameter) if we impose initial conditions: $T_{i,j,0} = 1$ and consider only $k > 0$.

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- This is also a cluster algebra mutation, and $T_{i,j,k}$ are cluster variables in an appropriate cluster algebra.

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The T-system as a non-commutative Q-system

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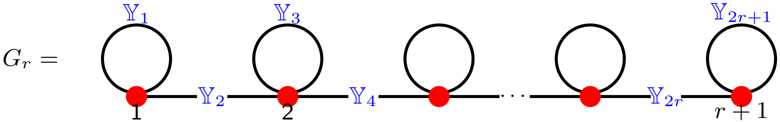
- This is an example of a non-commutative Q-system equation.

Solution of the T-system for A_r



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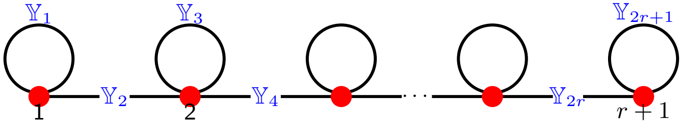
$G_r =$ 

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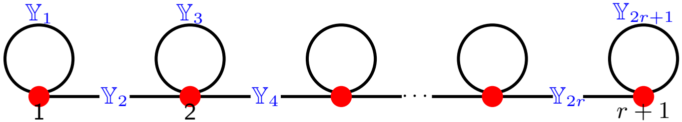
$G_r =$ 

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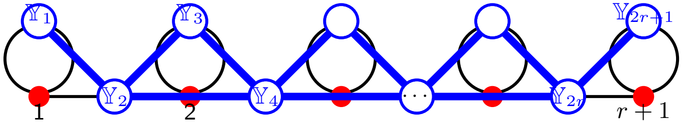
$G_r =$ 

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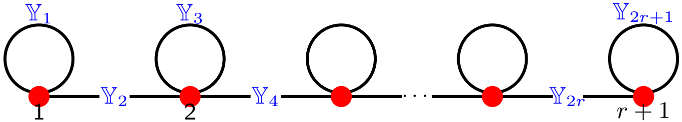
$G_r =$ 

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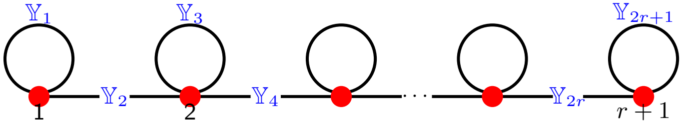

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Theorem (Di Francesco, K.)

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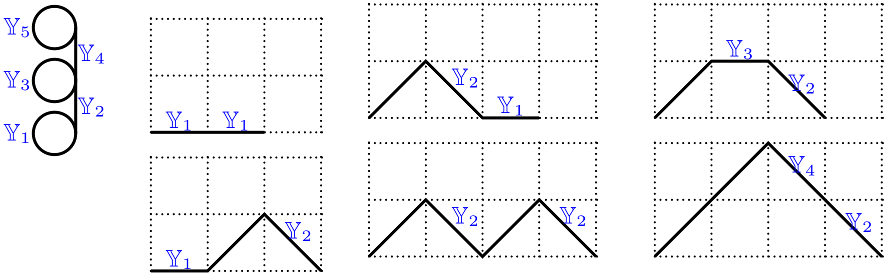
- **Linear recursion relation:** $\sum_{j=0}^{r+1} (-1)^j C_j \mathbb{T}_{1,k-j} = 0.$
- $\mathbb{T}_{1,k} \mathbb{T}_{1,0}^{-1} = (1 + Z_{1,1} \mathbb{Y}_1)[k]$ *(homogeneous component in* $\mathbb{Y}_i$ *of degree* k*).*

Example of non-commutative partition function

For A_2 we have the following paths contributing to $\mathbb{T}_{1,3}$ on the graph G_2



$$\mathbb{T}_{1,3}\mathbb{T}_{1,0}^{-1} = (1 + Z_{1,1}\mathbb{Y}_1)[3] = Z_{1,1}[2]\mathbb{Y}_1 = (\mathbb{Y}_1^2 + \mathbb{Y}_2\mathbb{Y}_1 + \mathbb{Y}_1\mathbb{Y}_2 + \mathbb{Y}_2^2 + \mathbb{Y}_3\mathbb{Y}_2 + \mathbb{Y}_4\mathbb{Y}_2)\mathbb{Y}_1.$$

Mutations of non-commutative weights

- There are many choices of initial conditions for the T-system: We limit ourselves to those described by Motzkin paths $\mathbf{m} = (m_1, ..., m_r)$. ($d^2$-invariant).

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- The variable $\mathbb{T}_{1,k+m_1}\mathbb{T}_{1,m_1}^{-1}$ as a function of $\mathbb{T}_{\mathbf{m}}$, the mutated data, is given by path partition function on G_r with new weights $\mathbb{Z}_i$

$$\mathbb{Z}_{2i} = (\mathbb{Y}_{2i+1})^{m_{i+1}-m_i}\mathbb{Y}_{2i}, \qquad \mathbb{Z}_{2i-1} = \mathbb{Y}_{2i-1} + \begin{cases} -\mathbb{Y}_{2i+1}^{-1}\mathbb{Y}_{2i}, & m_{i+1}-m_i = -1 \\ \mathbb{Y}_{2i}, & m_{i+1}-m_i = 1 \\ 0 & m_i - m_{i+1} = 0. \end{cases}$$

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- Rank 2 completely non-commutative case related to the “wall crossing formulas” of Kontsevich and Soibelman.