Differential equations for correlation functions in LFT, elliptic conformal blocks and AGT conjecture

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Liouville theory

- Lagrangian: $\mathcal{L} = \frac{1}{4\pi}(\partial_a \varphi)^2 + \mu e^{2b\varphi}$

- Central charge: $c_L = 1 + 6Q^2$ where $Q = b + \frac{1}{b}$

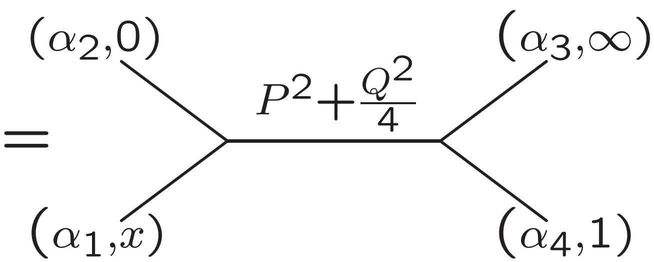
- Primary fields: $V_\alpha = e^{2\alpha\varphi}$ have conformal dimensions $\Delta(\alpha) = \alpha(Q-\alpha)$

- Three-point function (Dorn-Otto-Zamolodchikov-Zamolodchikov):

$$C(\alpha_1, \alpha_2, \alpha_3) = \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{\frac{(Q-\alpha)}{b}} \times$$
$$\times \frac{\Upsilon(b)\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\alpha - Q)\Upsilon(\alpha - 2\alpha_1)\Upsilon(\alpha - 2\alpha_2)\Upsilon(\alpha - 2\alpha_3)},$$

- Four-point function: $\langle V_{\alpha_1}(z_1,\bar{z}_1)V_{\alpha_2}(z_2,\bar{z}_2)V_{\alpha_3}(z_3,\bar{z}_3)V_{\alpha_4}(z_4,\bar{z}_4)\rangle \sim$

$$\sim \int_{\mathcal{C}} C\Big(\alpha_1,\alpha_2,\frac{Q}{2}+iP\Big)C\Big(\frac{Q}{2}-iP,\alpha_3,\alpha_4\Big)\left|\mathfrak{F}_P\begin{pmatrix}\alpha_2 & \alpha_3\\ \alpha_1 & \alpha_4\end{pmatrix}\!\Big| x\Big)\right|^2 dP,$$

- Conformal block: $\mathfrak{F}_P\begin{pmatrix}\alpha_2 & \alpha_3\\ \alpha_1 & \alpha_4\end{pmatrix}\!\Big| x\Big) =$ 

is not known in a closed form

- Elliptic block (Al. Zamolodchikov):

$$\mathfrak{F}_P\begin{pmatrix}\alpha_2 & \alpha_3\\ \alpha_1 & \alpha_4\end{pmatrix}\!\Big| x\Big) = (16q)^{P^2} x^{\frac{Q^2}{4}-\Delta_1-\Delta_2}(x-1)^{\frac{Q^2}{4}-\Delta_1-\Delta_4}\times$$
$$\times\,\theta_3(q)^{3Q^2-4\sum_k \Delta_k}\mathfrak{H}_P\begin{pmatrix}\alpha_2 & \alpha_3\\ \alpha_1 & \alpha_4\end{pmatrix}\!\Big| q\Big),$$

where $q=e^{i\pi\tau}$ with $\tau = i\frac{K(1-x)}{K(x)}$, satisfies a recursive relation which leads to an effective algorithm for calculation of its expansion in power series of q (which is more convenient for numerical studies than the ordinary x expansion)

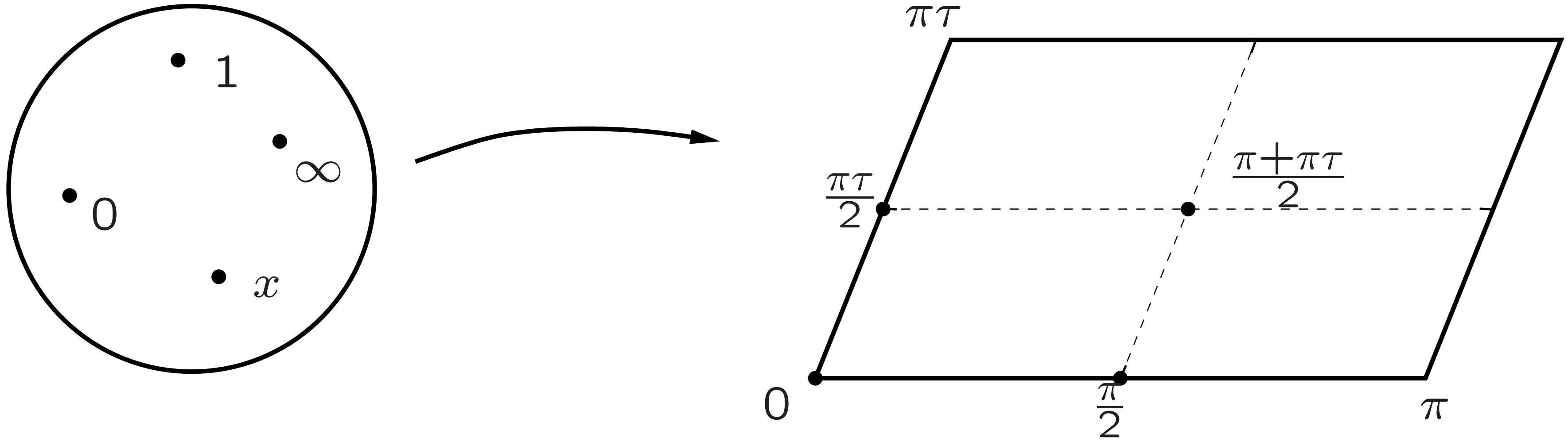
- Degenerate fields V_α with $\alpha = \alpha_{mn} = -\frac{mb}{2} - \frac{n}{2b}$ have a null-vector in their Verma module at level $(m+1)(n+1)$ and hence four-point function satisfies Fuchsian ordinary differential equation of the same order (Belavin-Polyakov-Zamolodchikov 1984). An explicit integral representaion for the solution to this equation can be obtained. For example in the case $n=0$ one has

$$\langle V_{-\frac{mb}{2}}(x,\bar{x}) V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(\infty)\rangle = \Omega_m(\alpha_1,\alpha_2,\alpha_3)\, |x|^{2mb\alpha_1}|x-1|^{2mb\alpha_2}$$
$$\times \int \prod_{k=1}^{m} |t_k|^{2A}|t_k-1|^{2B}|t_k-x|^{2C} \prod_{i<j} |t_i-t_j|^{-4b^2}\, d^2t_1 \dots d^2t_m$$

with parameters

$$A = b\,(\alpha - 2\alpha_1 - Q + mb/2)\,, \quad B = b\,(\alpha - 2\alpha_2 - Q + mb/2)\,,$$
$$C = b\,(Q + mb/2 - \alpha)$$

- However, for several important purposes one needs the differential operator for the four-point correlation function in explicit form.



- We consider five-point function

$$\langle V_{-\frac{1}{2b}}(z)V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)V_{\alpha_4}(x)\rangle =$$

$$= z^{\frac{1}{2b^2}}(z-1)^{\frac{1}{2b^2}}\frac{(z(z-1)(z-x))^{\frac{1}{4}}}{(x(x-1))^{\frac{2\Delta(\alpha_4)}{3}+\frac{1}{12}}}\frac{\Theta_1(u)^{b^{-2}}}{\Theta_1'(0)^{\frac{b^{-2}+1}{3}}}\,\Psi(u|q),$$

with $q = e^{i\pi\tau}$

$$u = \frac{\pi}{4K(x)}\int_0^{\frac{z-x}{x(z-1)}}\frac{dt}{\sqrt{t(1-t)(1-xt)}} \quad \text{and} \quad \tau = i\,\frac{K(1-x)}{K(x)}.$$

- One finds, that $\Psi(u|\tau)$ satisfies:

$$\left[\partial_u^2 - \mathbb{U}(u|\tau) + \frac{4i}{\pi b^2}\partial_\tau\right]\Psi(u|\tau) = 0, \qquad (*)$$

$$\mathbb{U}(u|\tau) = \sum_{j=1}^{4} s_j(s_j+1)\wp(u-\omega_j)$$

parameters s_k are related with α_k as

$$\alpha_k = \frac{Q}{2} - \frac{b}{2}\left(s_k + \frac{1}{2}\right)$$

and ω_k are half periods.

- One can try to find a solution to (*) in a form

$$\Psi(u|q) = u^{s_4+1}\left(\Psi(\tau) + \Psi_{-1}(\tau)u^2 + \Psi_{-2}(\tau)u^4 + \dots\right),$$

with diagonal monodromy around $u = 0$.

- Function $\vec{\Psi}(\tau) = \begin{pmatrix} \cdots \\ \Psi_{-1}(\tau) \\ \Psi(\tau) \end{pmatrix}$ satisfies semi-infinite WZ equation

$$\left(-J_- + \frac{i}{\pi b^2}\frac{\partial}{\partial\tau} + \sum_{k=1}^{\infty} \frac{W^{(k+1)}(\tau)}{k!^2} J_+^k\right)\vec{\Psi}(\tau) = 0, \qquad (**)$$

where

$$J_- = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & 0 & 0 \\ \cdots & -2s_4 - 5 & 0 & 0 \\ \cdots & 0 & -s_4 - \frac{3}{2} & 0 \end{pmatrix} \qquad J_+ = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & 1 & 0 \\ \cdots & 0 & 0 & 1 \\ \cdots & 0 & 0 & 0 \end{pmatrix}$$

and

$$W^{(k)}(\tau(x)) = \left(\frac{2K(x)}{\pi}\right)^{2k} \Big((-1)^{k+1}(x-1)\mathsf{w}_1^{(k)} P_k(x) +$$
$$+ x\mathsf{w}_2^{(k)} P_k(1-x) - (-1)^{k-1} x(x-1)\mathsf{w}_3^{(k)} x^{k-2} P_k(1/x)\Big)$$

with $\mathsf{w}_j^{(k)} = \left[s_j(s_j+1) + \frac{s_4(s_4+1)}{(4^k-1)}\right]$ and $\mathrm{sn}^2(t|\sqrt{x}) = \sum \frac{P_{k+1}(1-x)}{k!^2} t^{2k}$.

- One can notice that if the parameter s_4 in eq (**) from the previous slide takes the values*

$$s_4 = -m - \frac{3}{2}$$

then the infinite chain of equations (**) has a finite sub-chain. Due to the triangle form of (**) it is easy to conclude that the function $\Psi(\tau)$ satisfies a differential equation of the order $(m+1)$. Examples are (here $W^{(k)}(x) = (\frac{d\tau}{dx})^k W^{(k)}(\tau)$)

  - For $m = 1$ $$\Big(\partial^2 + W^{(2)}(x)\Big)\,\Psi = 0$$
  - For $m = 2$ $$\Big(\partial^3 + 4W^{(2)}(x)\partial + 2\partial W^{(2)}(x) + W^{(3)}(x)\Big)\,\Psi = 0$$
  - For $m = 3$ $$\Big(\partial^4 + 10W^{(2)}(x)\partial^2 + \Big(10\partial W^{(2)}(x) + 6W^{(3)}(x)\Big)\partial + \\ +\Big(9W^{(2)}(x)^2 + 3\partial^2 W^{(2)}(x) + 3\partial W^{(3)}(x) + W^{(4)}(x)\Big)\Big)\Psi = 0$$

*It corresponds to the situation $\alpha_4 = \frac{1}{2b} - \frac{mb}{2}$ and hence in the operator product expansion $V_{-\frac{1}{2b}}(z)V_{\alpha_4}(x)$ appears the degenerate field $V_{-\frac{mb}{2}}$.

Integrable potentials and conformal blocks

- We consider again the generalized Lamé heat equation

$$\left[\partial_u^2 - \mathbb{U}(u|\tau) + \frac{4i}{\pi b^2}\partial_\tau\right]\Psi(u|\tau) = 0, \qquad (\star)$$

with

$$\mathbb{U}(u|\tau) = \sum_{j=1}^{4} s_j(s_j+1)\wp(u-\omega_j)$$

- We propose that for $s_k = m_k + \frac{2n_k}{b^2}$ $(m_k, n_k \in \mathbb{Z}_+)$ equation is integrable
- For example for $s_1 = s_2 = s_3 = 0$ and $s_4 = 1$ one can construct explicit solution to $(\star)$

$$\Psi(u|q) = \int_0^\pi \left(\frac{\Theta_1(v)}{\Theta_1'(0)^{\frac{1}{3}}}\right)^{b^2} \frac{E(u+v)}{E(u)E(v)}\Psi_0(u+b^2v|q)\,dv,$$

where Ψ_0 is the solution of the heat equation and $E(u) = \frac{\Theta_1(u)}{\Theta_1'(0)}$

- In the dual case $s_1 = s_2 = s_3 = 0$ and $s_4 = \frac{2}{b^2}$

$$\Psi(u|q) = \Theta_1'(0)^{\frac{2}{3}(1-\frac{2}{b^2})} \int_0^\pi \left(\frac{\Theta_1(v)}{\Theta_1'(0)^{\frac{1}{3}}}\right)^{\frac{4}{b^2}} \left(\frac{E(u+v)}{E(u)E(v)}\right)^{\frac{2}{b^2}} \Psi_0(u+2v|q)\, dv,$$

- For general $s_k = m_k + \frac{2n_k}{b^2}$, we expect solution is likely to be given by an integral of dimension

$$N = g + n_1 + n_2 + n_3 + n_4,$$

where g is the number of gaps for the classical potential

$$g = \frac{1}{2}\Big(2\max m_k, 1 + \mathtt{m} - (1+(-1)^{\mathtt{m}})\Big(\min m_k + \frac{1}{2}\Big)\Big),$$

here $\mathtt{m} = \sum m_k$. For example, for $s_1 = s_2 = s_3 = 0$ and $s_4 = m$

$$\Psi(u|q) = \int_0^\pi .. \int_0^\pi \prod_{k=1}^m \left(\frac{\Theta_1(v_k)}{\Theta_1'(0)^{\frac{1}{3}}}\right)^{mb^2} \prod_{i<j} \left|\frac{\Theta_1(v_i - v_j)}{\Theta_1'(0)^{\frac{1}{3}}}\right|^{-b^2}$$
$$\prod_{k=1}^m \frac{E(u+v_k)}{E(u)E(v_k)} \Psi_0(u + b^2 v|q)\, dv_1 ... dv_m,$$

- In order to obtain the conformal block one has to take instead of Ψ_0

$$\Psi_P^{\pm}(u|q) = q^{P^2} e^{\pm 2b^{-1}Pu}.$$

and take the limit $u \to 0$

- Let us define:

$$\mathcal{H}_m^{(P)}(q) \stackrel{\text{def}}{=} \mathfrak{H}_P\left(\begin{matrix} \frac{Q}{2} - \frac{b}{4} & \frac{Q}{2} - \frac{b}{4} \\ -\frac{(2m-1)b}{4} & \frac{Q}{2} - \frac{b}{4} \end{matrix}\middle| q\right)$$

then

$$\mathcal{H}_m^{(P)}(q) = N_m^{-1} \int_0^\pi \cdots \int_0^\pi e^{2bP(u_1+\cdots+u_m)} \prod_{k=1}^m E(u_k)^{mb^2} \prod_{i<j} |E(u_i - u_j)|^{-b^2} \, d\vec{u}$$

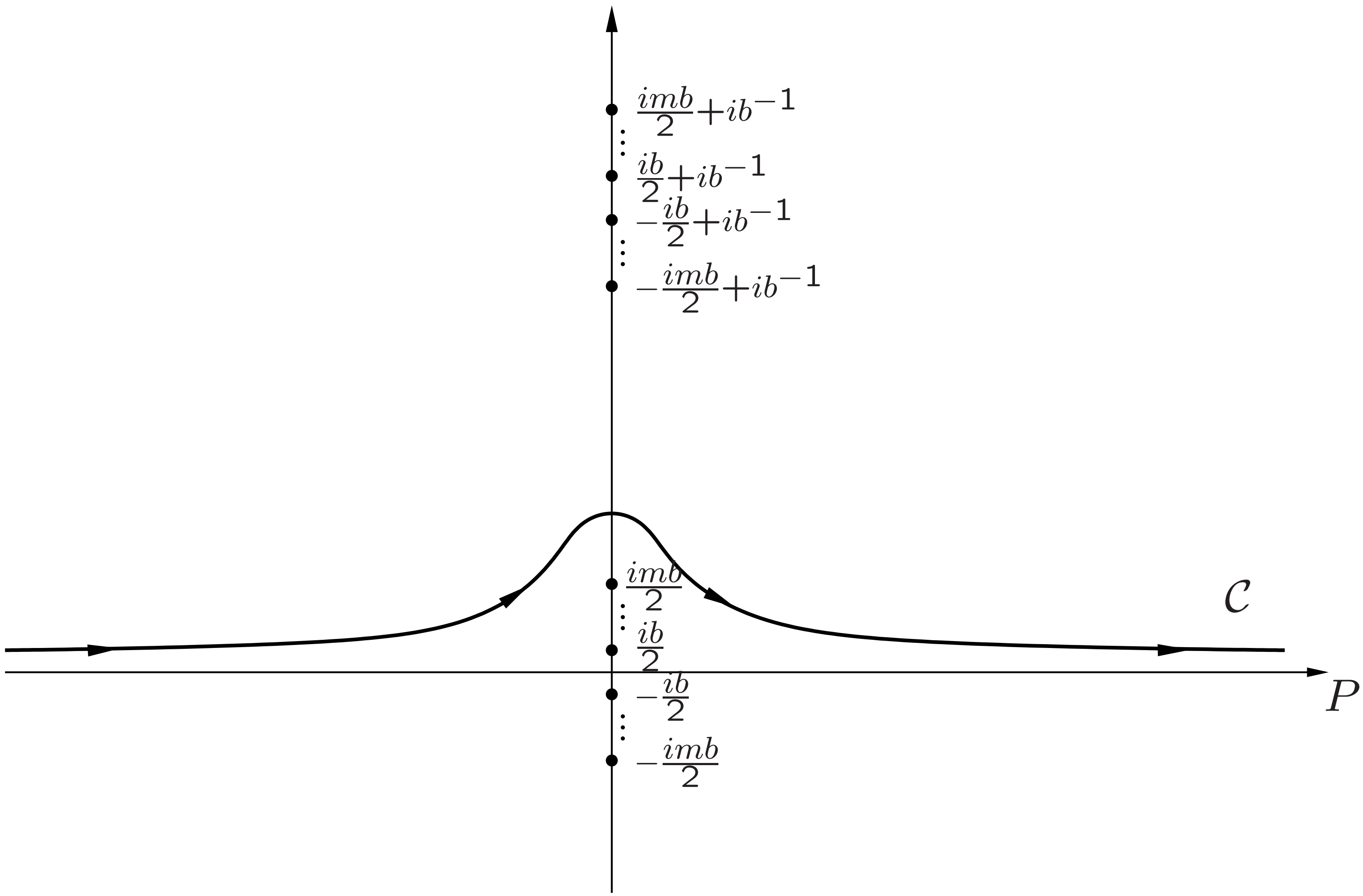
where N_m is the normalization constant

- The product of structure constants simplifies drastically

$$C\left(-\frac{(2m-1)b}{4}, \frac{Q}{2} - \frac{b}{4}, \frac{Q}{2} + iP\right) C\left(\frac{Q}{2} - iP, \frac{Q}{2} - \frac{b}{4}, \frac{Q}{2} - \frac{b}{4}\right) \sim$$

$$\sim 16^{-2P^2} \prod_{k=1}^m \gamma\left(ibP - \frac{kb^2}{2}\right) \gamma\left(-ibP - \frac{kb^2}{2}\right).$$

- The integral over the intermediate momentum P goes as shown



- This deformation of the contour is prescribed by the condition that the four-point correlation function is single-valued

- Surprisingly, the result of integration over the momentum P is given by a multiple integral over the torus T with periods π and $\pi\tau$

$$\int_{\mathcal{C}} \frac{|q|^{2P^2}\mathfrak{F}_m(P|\tau)\mathfrak{F}_m(-P|\tau^*)}{\prod_{k=1}^{m} \sin\left(\pi(ibP+\frac{kb^2}{2})\right)\sin\left(\pi(ibP-\frac{kb^2}{2})\right)}\,dP =$$

$$= \Lambda_m\Big(\mathrm{Im}(\tau)\Big)^{-1/2}\int_T\!\!\dots\!\!\int_T \prod_{k=1}^{m}\mathcal{E}(u_k,\bar{u}_k)^{mb^2}\prod_{i<j}\mathcal{E}(u_i-u_j,\bar{u}_i-\bar{u}_j)^{-b^2}\,d^2\vec{u},$$

where

$$\mathfrak{F}_m(P|\tau) \stackrel{\text{def}}{=} \int_0^\pi\!\!\dots\!\!\int_0^\pi e^{2bP(u_1+\dots+u_m)}\prod_{k=1}^{m}E(u_k)^{mb^2}\prod_{i<j}|E(u_i-u_j)|^{-b^2}\,d\vec{u},$$

$$\mathcal{E}(u,\bar{u}) = E(u)\bar{E}(\bar{u})e^{-\frac{2(\mathrm{Im}u)^2}{\pi\mathrm{Im}\tau}}$$

- We note that this integral representation looks like Coulomb gas representation of the one-point correlation function of the operator $V_{-mb'}$ in LFT with parameter $b' = \frac{b}{\sqrt{2}}$ on a torus

- Let us define function $\mathcal{T}(\alpha, b|q)$ in Liouville field theory with cosmological constant μ and coupling constant b on a torus

$$\mathcal{T}(\alpha, b|q) \stackrel{\text{def}}{=} \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{\frac{\alpha}{b}} |\eta(\tau)|^{-4\Delta(\alpha)}\langle V_\alpha\rangle_\tau$$

We define also the function $\mathcal{S}(\alpha, b|q)$ which is related to the four-point correlation function in LFT on sphere as (here $\zeta = \frac{Q}{2} - \frac{b}{4}$)

$$\mathcal{S}(\alpha, b|q) \stackrel{\text{def}}{=} \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{\frac{\alpha}{b}+\frac{1}{2b}-\frac{1}{4}} \times$$

$$\times\, |x(x-1)|^{\frac{4}{3}\Delta(\alpha)} \langle V_\alpha(x,\bar{x})V_\zeta(0)V_\zeta(1)V_\zeta(\infty)\rangle.$$

- The correspondence between the one-point toric and the four-point spheric correlation functions states that

$$\mathcal{S}(\alpha, b|q) = \aleph\left(\left(\alpha - \frac{b}{4}\right)\sqrt{2}, \frac{b}{\sqrt{2}}\right) \mathcal{T}\left(\left(\alpha - \frac{b}{4}\right)\sqrt{2}, \frac{b}{\sqrt{2}}\middle| q\right),$$

where $\aleph(\alpha, b)$ is given by

$$\aleph(\alpha, b) = \frac{\Upsilon_b(\alpha)}{\Upsilon_b\left(\frac{1}{2b}\right)} \frac{\Upsilon_b\left(\frac{1}{b}\right)}{\Upsilon_b\left(\alpha + \frac{1}{2b}\right)}.$$

Conformal blocks and Nekrasov partition function

One-point conformal block $\mathcal{F}_\alpha^{(\Delta)}(q)$ is defined as the contribution to the trace of the conformal family with conformal dimension $\Delta = \frac{Q^2}{4} + P^2$

$$\mathcal{F}_\alpha^{(\Delta)}(q) \stackrel{\text{def}}{=} \text{Tr}_\Delta \left(q^{L_0 - \frac{c}{24}} V_\alpha(0) \right) = 1 + \frac{2\Delta + \Delta^2(\alpha) - \Delta(\alpha)}{2\Delta} q + \ldots$$

It was proposed by Alday, Gaiotto and Tachikawa that

$$\mathcal{F}_\alpha^{(\Delta)}(q) = \left(\frac{q^{\frac{1}{24}}}{\eta(\tau)} \right)^{2\Delta(\alpha)-1} Z(\varepsilon_1, \varepsilon_2, m, a),$$

where $Z(\varepsilon_1, \varepsilon_2, m, a)$ is the instanton part of the Nekrasov partition function in $\mathcal{N} = 2^*$ $U(2)$ SYM with

$$P = \frac{a}{\hbar}, \qquad \alpha = \frac{m}{\hbar}, \qquad \varepsilon_1 = \hbar b, \qquad \varepsilon_2 = \frac{\hbar}{b},$$

where a is VEV of scalar field, m is the mass of the adjoint hypermultiplet and ε_k are the parameters of the Ω background. Parameter q is given by

$$q = e^{2i\pi\tau}, \quad \text{where} \quad \tau = \frac{4i\pi}{g^2} + \frac{\theta}{2\pi}.$$

Nekrasov partition function

$$Z(\varepsilon_1, \varepsilon_2, m, a) = 1 + \sum_{k=1}^{\infty} q^k \mathfrak{Z}_k,$$

can be represented as a sum over partitions. Let $\vec{Y} = (Y_1, Y_2)$ be the pair of Young diagrams with the total numbers of cells equal to N. Then

$$\mathfrak{Z}_N = \sum_{\vec{Y}} \prod_{i,j=1}^{2} \prod_{s \in Y_i} \frac{(E_{ij}(s) - \alpha)(Q - E_{ij}(s) - \alpha)}{E_{ij}(s)(Q - E_{ij}(s))},$$

where

$$E_{ij}(s) = 2P\epsilon_{ij} - bH_{Y_j}(s) + b^{-1}(V_{Y_i}(s) + 1),$$

$H_Y(s)$ and $V_Y(s)$ are respectively the horizontal and vertical distance from the square s to the edge of the diagram Y.

- AGT relation for $N = 2^*$ theory can proved using Al. Zamolodchikov's recursive formula

- The coefficient $\mathfrak{Z}_N$ can be represented as the contour integral

$$\mathfrak{Z}_N = \frac{1}{N!}\left(\frac{Q(b-\alpha)(b^{-1}-\alpha)}{2\pi\mathrm{i}\alpha(Q-\alpha)}\right)^N \oint_{\mathcal{C}_1}\dots\oint_{\mathcal{C}_N} \prod_{k=1}^{N} \frac{\mathcal{P}(x_k+\alpha)\mathcal{P}(x_k+Q-\alpha)}{\mathcal{P}(x_k)\mathcal{P}(x_k+Q)} \times$$
$$\times \prod_{i<j} \frac{x_{ij}^2(x_{ij}^2-Q^2)(x_{ij}^2-(b-\alpha)^2)(x_{ij}^2-(b^{-1}-\alpha)^2)}{(x_{ij}^2-b^2)(x_{ij}^2-b^{-2})(x_{ij}^2-\alpha^2)(x_{ij}^2-(Q-\alpha)^2)}\, dx_1\dots dx_N,$$

where $\mathcal{P}(x) = (x-P_1)(x-P_2)$ with $P = (P_1-P_2)/2$. The contour $\mathcal{C}_k$ surrounds poles $x_k = P_1$, $x_k = P_2$, $x_k = x_j + b$ and $x_k = x_j + b^{-1}$.

- A singularity in $\mathfrak{Z}_N = \mathfrak{Z}_N(\alpha,\Delta)$ ($\Delta = Q^2/4 + P^2$) can happen when two poles of the integrand pinch the contour. One can show that

$$\mathrm{Res}\, \mathfrak{Z}_N(\alpha,\Delta)\Big|_{\Delta=\Delta_{m,n}} = R_{m,n}(\alpha)\, \mathfrak{Z}_{N-mn}(\alpha,\Delta_{m,-n}),$$

where $R_{m,n}(\alpha)$ is exactly the same as prescribed by Alyosha Zamolodchikov's recursion formula.

- So, the singular part of the Nekrasov partition function coincides with the singular part of the one-point conformal block.

- The non-singular part which can be obtained in the limit $\Delta \to \infty$. It can be found using well known "hook-length" formula

$$\left(\frac{q^{\frac{1}{24}}}{\eta(\tau)}\right)^{1-\lambda} = 1 + \sum_{k=1}^{\infty} \xi_k(\lambda) q^k,$$

with

$$\xi_N(\lambda) = \sum_{Y} \prod_{s\in Y} \left(1 - \frac{\lambda}{\left(1 + H_Y(s) + V_Y(s)\right)^2}\right).$$

the sum goes over all Y's with the total number of cells equal to N.

- Comparing asymptotics of the conformal block and Nekrasov partition function one finds the coefficient of proportionality in AGT formula.

- Seiberg-Witten prepotential can be obtained in the semiclassical limit $\hbar \to 0$

$$Z(\varepsilon_1, \varepsilon_2, m, \vec{a}) \to \exp\left(\frac{1}{\hbar^2}\mathcal{F}\left(m, \vec{a}|q\right) + O(1)\right).$$

- To derive this limit from the Liouville point of view we consider two-point function with one degenerate field

$$\Psi(z) \sim \langle V_{-\frac{b}{2}}(z) V_\alpha(0)\rangle$$

This function satisfies Scrödinger equation

$$\left(-\partial_z^2 + \frac{b^2 m^2}{\hbar^2}\wp(z)\right)\Psi(z) = \frac{2ib^2}{\pi}\partial_\tau \Psi(z).$$

- We look for the solution in the form

$$\Psi(z) = \exp\left(\frac{1}{\hbar^2}\mathcal{F}(q) + \frac{b}{\hbar}\mathcal{W}(z|q) + \dots\right)$$

with prescribed monodromy $e^{2i\pi a}$ around A-cycle.

- WKB approximation gives

$$\mathcal{W}(z|q) = \int \sqrt{E(q) + m^2 \wp(z)} dz, \qquad E(q) = 4q\partial_q \mathcal{F}(q).$$

- With $E(q)$ given in parametric form

$$\oint_A \sqrt{E(q) + m^2 \wp(z)} dz = 2i\pi a,$$

the prepotential $\mathbb{F}(m, \vec{a}|q)$ can be calculated as follows

$$\mathbb{F}(m, \vec{a}|q) = \left(a^2 + \frac{m^2}{12}\right) \log(q) - 4m^2 \log(\eta(\tau)) + \mathcal{F}(q),$$

- The integral over B-cycle defines a_D

$$\oint_B \sqrt{E(q) + m^2 \, \wp(z)} \, dz = 2i\pi a_D,$$

which is the derivative of the total prepotential (including classical and perturbative part) with respect to a.