The quantum N-body problem and the auxiliary field method

Claude Semay

claude.semay@umons.ac.be

F.R.S.-FNRS Senior Research Associate

University of Mons - UMONS

30 years of strong interactions – Spa 2011

Collaboration

University of Mons

- C. Semay
- F. Buisseret
- F. Brau

LPSC Grenoble

- B. Silvestre-Brac

Outline

The auxiliary field method (AFM):
to obtain **approximate closed-form** solutions for eigenequations in quantum mechanics

Useful theorems

The Hellmann-Feynman theorem

$H(\lambda)$ is the Hamiltonian and λ is a parameter

$$H(\lambda)|\lambda\rangle = E(\lambda)|\lambda\rangle$$

$E(\lambda)$ is the energy and $|\lambda\rangle$ the normalized bound eigenstate

$$\frac{\partial E(\lambda)}{\partial \lambda} = \left\langle \lambda \left| \frac{\partial H(\lambda)}{\partial \lambda} \right| \lambda \right\rangle$$

Example: $\dfrac{\partial E}{\partial m_i} < 0$ for a N-body nonrelativistic system with interaction independent of m_i

D. B. Lichtenberg, Phys. Rev. D **40**, 4196 (1989)

Useful theorems

The general virial theorem

H is a two-body Hamiltonian

$$H = T(\mathbf{p}) + V(\mathbf{r}) + K$$

$|\phi\rangle$ is a normalized eigenstate

$$\langle \phi | \mathbf{p}\nabla_{\mathbf{p}} T(\mathbf{p}) | \phi \rangle = \langle \phi | \mathbf{r}\nabla_{\mathbf{r}} V(\mathbf{r}) | \phi \rangle$$

Example:

$$H = 2\sqrt{\mathbf{p}^2} + a\,r \quad \rightarrow \quad \left\langle \phi \left| 2\sqrt{\mathbf{p}^2} \right| \phi \right\rangle = \langle \phi | a\,r | \phi \rangle = M/2$$

W. Lucha, Mod. Phys. Lett. A **5**, 2473 (1990)

Useful theorems

The comparison theorem

$$\begin{aligned} H^{(1)} &= T(\mathbf{p}) + V^{(1)}(\mathbf{r}) \\ H^{(2)} &= T(\mathbf{p}) + V^{(2)}(\mathbf{r}) \end{aligned}$$

We assume that $E^{(1)}_{\{\alpha\}}$ and $E^{(2)}_{\{\alpha\}}$ exist

where $E^{(i)}_{\{\alpha\}}$ is an eigenvalue of $H^{(i)}$ with quantum number $\{\alpha\}$

$$V^{(2)} \geq V^{(1)} \quad \Rightarrow \quad E^{(2)}_{\{\alpha\}} \geq E^{(1)}_{\{\alpha\}}$$

Example:

If $V(\mathbf{r}) = \kappa v(r)$ with $v(r) > 0$, then eigenvalues increase with κ

C. Semay, Phys. Rev. A **83**, 024101 (2011)

Nonrelativistic two-body systems

The auxiliary field $\hat\nu$

We want to study

$$H = \frac{\mathbf{p}^2}{2m} + V(r)$$

Let us consider a new Hamiltonian

$$\tilde H(\hat\nu) = \frac{\mathbf{p}^2}{2m} + \tilde V(r,\hat\nu) \quad \text{with}$$

$$\tilde V(r,x) = x\,P(r) + V\left(I(x)\right) - x\,P\left(I(x)\right),$$

$$I(x) = K^{-1}(x) \quad \text{and} \quad K(x) = \frac{V'(x)}{P'(x)}$$

Variational procedure:

$$\delta_{\hat\nu}\tilde H(\hat\nu)\Big|_{\hat\nu=\hat\nu_0} = 0 \Rightarrow \hat\nu_0 = K(r) \text{ and } \tilde H(\hat\nu_0) = H$$

Nonrelativistic two-body systems

The approximation

To replace $\hat{\nu}$ by a real parameter ν
$\Rightarrow \tilde{V}(r,\nu)$ is a linear function of $P(r)$

For a good choice of $P(r)$, $\tilde{H}(\nu)$ is solvable and

$$E(\nu) = e(\nu) + V\left(I(\nu)\right) - \nu\, P\left(I(\nu)\right)$$

where

$$h(\nu)\,|\psi(\nu)\rangle = e(\nu)\,|\psi(\nu)\rangle \quad \text{with} \quad h(\nu) = \frac{\mathbf{p}^2}{2m} + \nu\, P(r)$$

Approximate solutions given by $E(\nu_0)$ and $|\psi(\nu_0)\rangle$:

$$\partial_\nu E(\nu)\big|_{\nu=\nu_0} = 0$$

Nonrelativistic two-body systems

Useful formulas

$e(\nu)$ is an eigenvalue of $\mathbf{p}^2/(2m) + \nu\, P(r)$

Definition of a mean radius $r_0 = I(\nu_0)$
with $I(x) = K^{-1}(x)$ and $K(x) = V'(x)/P'(x)$

$\partial_\nu E(\nu)|_{\nu=\nu_0} = 0$: transcendental equation for $r_0 = r_0(n, l)$

$$P(r_0) = e'(K(r_0)) \quad \text{with} \quad e'(\nu) = \partial_\nu e(\nu)$$

The AFM energy is

$$E_{\text{AFM}} = E(K(r_0)) = \underbrace{e(K(r_0)) - K(r_0)\, P(r_0)}_{\langle \psi(\nu_0)|\mathbf{p}^2/(2m)|\psi(\nu_0)\rangle} + V(r_0)$$

Nonrelativistic two-body systems

Approximate potential

$V(r)$ is approximated by

$$\begin{aligned}\tilde{V}(r,\nu_0) &= \nu_0\, P(r) + V\left(I(\nu_0)\right) - \nu_0\, P\left(I(\nu_0)\right) \\ &= \frac{V'(r_0)}{P'(r_0)}(P(r) - P(r_0)) + V(r_0)\end{aligned}$$

because $r_0 = I(\nu_0)$ and $K(r_0) = \nu_0$

$\tilde{V}(r,\nu_0)$ is tangent to $V(r)$ at r_0:

- $\tilde{V}(r_0,\nu_0) = V(r_0)$
- $\tilde{V}'(r_0,\nu_0) = V'(r_0)$

Nonrelativistic two-body systems

Upper and lower bounds - 1

Consequence of the comparison theorem:

- $\tilde{V}(r,\nu_0) \geq V(r)$: E_{AFM} is an upper bound
- $\tilde{V}(r,\nu_0) \leq V(r)$: E_{AFM} is a lower bound

Nonrelativistic two-body systems

Upper and lower bounds - 1

Consequence of the comparison theorem:

- $\tilde{V}(r,\nu_0) \geq V(r)$: E_{AFM} is an upper bound
- $\tilde{V}(r,\nu_0) \leq V(r)$: E_{AFM} is a lower bound

Upper and lower bounds - 2

Let us define $g(x)$ by $V(x) = g(P(x))$:

- $g(x)$ concave: E_{AFM} is an upper bound
- $g(x)$ convex: E_{AFM} is a lower bound

Nonrelativistic two-body systems

Power-law potentials

$V(r) = \mathrm{sgn}(\eta)\, a\, r^\eta$ with $a > 0,\ -1 \le \eta \le 2$

$$E_{\mathrm{AFM}} = \frac{2+\eta}{2\eta} (a|\eta|)^{\frac{2}{\eta+2}} \left(\frac{Q(n,l)^2}{m} \right)^{\frac{\eta}{\eta+2}}$$

- $Q(n,l) = 2n + l + 3/2$ for $P(r) = r^2$ (upper bound)
- $Q(n,0) = 2\,(-\alpha_n/3)^{3/2}$ for $P(r) = r$
  α_n: zero of the Airy function
- $Q(n,l) = n + l + 1$ for $P(r) = -1/r$ (lower bound)

Nonrelativistic two-body systems

Connection with envelope theory

Search for $\tilde{V}(r,\nu_0)$ is the basis of the envelope theory developed by Richard L. Hall ($\sim$ 1980)

Novelties in AFM: $\hat{\nu}_0 = K(r)$ and $\nu_0 = K(r_0)$

- $\langle\psi(\nu_0)|P(r)|\psi(\nu_0)\rangle = P(r_0)$
- $\langle\psi(\nu_0)|Z(\hat{\nu}_0)|\psi(\nu_0)\rangle = Z(\nu_0)$ with $Z(x) = P(I(x))$
  The method is a kind of mean field approximation
- $\langle\psi(\nu_0)|V(r)|\psi(\nu_0)\rangle = V(r_0) - E_{\mathrm{AFM}} + \langle\psi(\nu_0)|H|\psi(\nu_0)\rangle$
  For the ground state:
  $E_{\mathrm{AFM}} - E \geq V(r_0) - \langle\psi(\nu_0)|V(r)|\psi(\nu_0)\rangle$

Nonrelativistic two-body systems

General equations

$$T(x) = \frac{x^2}{2m}$$

$$E_{\text{AFM}} = T(p_0) + V(r_0) \quad \text{(semiclassical energy)}$$

$$p_0 = \frac{Q(n,l)}{r_0} \quad \text{(correspondence principle)}$$

$$\frac{p_0^2}{m} = r_0 V'(r_0) \quad \text{(virial theorem)}$$

$Q(n,l)$ depends on $P(r)$

OK for $V(x) = \alpha\, x^a + \beta\, x^b$, $\ln(\alpha\, x)$, $-k\, e^{-x^a}$, $\sqrt{\alpha\, x^2 + \beta}$, ...

Critical k for $V(x) = -k\, e^{-x^a}$, $-k\, \frac{e^{-x}}{x}$, ...

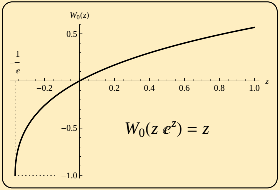
Nonrelativistic two-body systems

Exponential potentials

$$h = \mathbf{p}^2 - k\, e^{-r^\eta} \quad \text{with} \quad k > 0,\ \eta > 0$$

$$E_{\text{AFM}} = -k \left(\frac{-Y}{W_0(-Y)} \right)^{\frac{\eta+2}{\eta}} \left[1 + \frac{\eta+2}{2} W_0(-Y) \right]$$

with $Y = \frac{\eta}{\eta+2} \left(\frac{2\, Q(n,l)^2}{\eta\, k} \right)^{\frac{\eta}{\eta+2}}$ and



Critical k: $k_{\eta;nl} = \left(\frac{\eta\, e}{2} \right)^{\frac{2}{\eta}} Q(n,l)^2$

Semirelativistic kinematics

The auxiliary field $\hat{\mu}$ and parameter μ

We want to study ($\sigma = 1$ or 2)

$$H = \sigma \sqrt{\mathbf{p}^2 + m^2} + V(r)$$

Transformation into an equivalent nonrelativistic Hamiltonian

$$\tilde{H}(\mu) = \tilde{T}(\mathbf{p}^2, \mu) + V(r) \quad \text{with} \quad \tilde{T}(\mathbf{p}^2, \mu) = \frac{\mathbf{p}^2 + m^2}{\mu} + \frac{\sigma^2}{4}\mu$$

Extremal eigensolutions of $\tilde{H}(\mu)$:

- $M(\mu_0)$ is an upper bound
- $\mu_0^2 = \langle \psi(\mu_0) | \hat{\mu}^2 | \psi(\mu_0) \rangle$ with $\hat{\mu} = \frac{2}{\sigma}\sqrt{\mathbf{p}^2 + m^2}$

Semirelativistic kinematics

Two auxiliary parameters

- Auxiliary parameter μ for $\sigma\sqrt{\mathbf{p}^2+m^2}$
- Auxiliary parameter ν for $V(r)$ with $P(r)=\mathrm{sgn}(\lambda)\, r^\lambda$

$$
\begin{aligned}
\tilde{H}(\mu,\nu) &= \tilde{T}(\mathbf{p}^2,\mu)+\tilde{V}(r,\nu) \\
\tilde{T}(\mathbf{p}^2,\mu) &= \frac{\mathbf{p}^2+m^2}{\mu}+\frac{\sigma^2}{4}\mu \\
\tilde{V}(r,\nu) &= \mathrm{sgn}(\lambda)\,\nu\, r^\lambda + V(I(\nu)) - \mathrm{sgn}(\lambda)\nu I(\nu)^\lambda
\end{aligned}
$$

M_{AFM} is given by extremal solutions $M(\mu_0,\nu_0)$ of $\tilde{H}(\mu,\nu)$

Semirelativistic kinematics

Simplified formulation - 1

μ_0 and ν_0 can be expressed in terms of x_0

$$\begin{aligned} \mu_0(x_0) &= \frac{2}{\sigma}\sqrt{m^2 + Q^{2\lambda/(\lambda+2)} x_0} \\ \nu_0(x_0) &= K\left(Q^{2/(\lambda+2)}/\sqrt{x_0}\right) \end{aligned}$$

with

$$x_0 = \left(\frac{|\lambda|}{2}\mu_0\nu_0\right)^{2/(\lambda+2)}$$

The structure of Q depends on λ

Semirelativistic kinematics

Simplified formulation - 2

The mean radius is then given by

$$r_0 = \frac{Q^{2/(\lambda+2)}}{\sqrt{x_0}}$$

The unique transcendental equation for r_0 is

$$\sigma Q = r_0^2 V'(r_0) \sqrt{1 + \left(\frac{m r_0}{Q}\right)^2}$$

The AFM Mass is then

$$M_{\mathrm{AFM}} = \frac{\sigma Q}{r_0} \sqrt{1 + \left(\frac{m r_0}{Q}\right)^2} + V(r_0)$$

Semirelativistic kinematics

General equations

$T(x) = \sigma\sqrt{x^2 + m^2}$

$$M_{\mathrm{AFM}} = T(p_0) + V(r_0) \quad \text{(semiclassical energy)}$$

$$p_0 = \frac{Q(n,l)}{r_0} \quad \text{(correspondence principle)}$$

$$p_0\, T'(p_0) = r_0\, V'(r_0) \quad \text{(general virial theorem)}$$

$Q(n,l)$ depends on $P(r)$

OK for $V(x) = \alpha\, x^a + \beta\, x^b$, $\ln(\alpha\, x)$, $\sqrt{\alpha\, x^2 + \beta}$, ...

Critical k for $V(x) = -k\,\dfrac{e^{-x}}{x}$, ...

Semirelativistic kinematics

Power-law potentials

$$H = \sigma\,\sqrt{\mathbf{p}^2 + m^2} + \mathrm{sgn}(\eta)\, a\, r^\eta$$

- $m \neq 0$: analytical solutions exist for some η
- $m = 0$: $M_{\mathrm{AFM}} = \frac{\eta+1}{\eta}\,(a|\eta|)^{\frac{1}{\eta+1}}\,(\sigma Q)^{\frac{\eta}{\eta+1}} \qquad (\eta > 0)$

Semirelativistic kinematics

Power-law potentials

$$H = \sigma\sqrt{\mathbf{p}^2 + m^2} + \mathrm{sgn}(\eta)\, a\, r^\eta$$

- $m \neq 0$: analytical solutions exist for some η
- $m = 0$: $M_{\mathrm{AFM}} = \frac{\eta+1}{\eta}\,(a|\eta|)^{\frac{1}{\eta+1}}\,(\sigma Q)^{\frac{\eta}{\eta+1}} \qquad (\eta > 0)$

Funnel potential

$$H = \sigma\sqrt{\mathbf{p}^2 + m^2} + a\, r - \frac{b}{r} \quad (a > 0, b \geq 0)$$

- $m \neq 0$: analytical solution exists
- $m = 0$: $M_{\mathrm{AFM}} = 2\sqrt{a(\sigma\, Q - b)}$

Semirelativistic kinematics

Duality relation

$$H_{\mathrm{NR}} = \frac{\mathbf{p}^2}{2m} + V(r) \quad \Rightarrow \quad E(m;Q)$$

$$H_{\mathrm{UR}} = \sigma\sqrt{\mathbf{p}^2} + W(r) \quad \Rightarrow \quad M(\sigma;Q)$$

$$E(m;Q) = M(\sigma;Q) \quad \text{if} \quad \begin{cases} V(\alpha\sqrt{r}) = W(r) \\ 2\alpha^2 m\sigma = Q \end{cases}$$

- Relations exact for the AFM approximations
- If $V(r) \propto r^2$, errors around 10–35% for exact solutions

N-body systems

General Hamiltonian

$$H = \sum_{i=1}^{N} \sqrt{\mathbf{p}_i^2 + m_i^2} + \sum_{i=1}^{N} V_i(|\mathbf{r}_i - \mathbf{R}|) + \sum_{i<j=1}^{N} \bar{V}_{ij}(|\mathbf{r}_i - \mathbf{r}_j|)$$

- $\sum_{i=1}^{N} \mathbf{p}_i = \mathbf{0}$
- $\mathbf{R}$ is the (nonrelativistic) cm
- $V_i(x)$: one-body potential
- $\bar{V}_{ij}(x)$: two-body potential

N-body systems

AFM Hamiltonian

$$
\begin{aligned}
\tilde{H} \;=\; & \sum_{i=1}^{N}\left[\frac{\mathbf{p}_i^2+m_i^2}{2\mu_i}+\frac{\mu_i}{2}\right] \\
& + \sum_{i=1}^{N}\left[\nu_i\, P(r_i) + V_i(I_i(\nu_i)) - \nu_i\, P(I_i(\nu_i))\right] \\
& + \sum_{i<j=1}^{N}\left[\bar{\nu}_{ij}\, \bar{P}(r_{ij}) + \bar{V}_{ij}(\bar{I}_{ij}(\bar{\nu}_{ij})) - \bar{\nu}_{ij}\, \bar{P}(\bar{I}_{ij}(\bar{\nu}_{ij}))\right]
\end{aligned}
$$

where $r_i = |\mathbf{r}_i - \mathbf{R}|$, $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$
and $I_i(x) = K_i^{-1}(x)$, $K_i(x) = V_i'(x)/P'(x)$, …

N-body systems

N-body harmonic oscillator: $P(x) = \bar{P}(x) = x^2$

$$H_{\text{ho}} = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m_i} + \sum_{i=1}^{N} k_i(\mathbf{r}_i - \mathbf{R})^2 + \sum_{i<j=1}^{N} \bar{k}_{ij}(\mathbf{r}_i - \mathbf{r}_j)^2$$

Eigenenergies given by

$$E_{\text{ho}} = \sum_{i=1}^{N-1} \omega_i(2n_i + l_i + 3/2)$$

ω_i are eigenvalues of an analytical $(N-1)$th order matrix

- Analytical expression for $N \leq 5$
- New result: $k_i \neq 0$

N-body systems

Identical particles

$m_i = m$, $k_i = k$ and $\bar{k}_{ij} = \bar{k}$:

$$E_{\text{ho}} = \sqrt{\frac{2}{m}(k + N\bar{k})}\; Q$$

where Q is the total principal quantum number

$$Q = \sum_{i=1}^{N-1} (2n_i + l_i) + \frac{3}{2}(N-1)$$

N-body systems

General Hamiltonian with identical particles

$m_i = m$, $V_k(x) = V(x)$ and $\bar{V}_{kl}(x) = \bar{V}(x)$

R is the cm

For (anti)symmetrical states: $\mu_i = \mu$, $\nu_i = \nu$ and $\bar{\nu}_{ij} = \bar{\nu}$

$$\begin{aligned}
M_{\mathrm{AFM}} &= N\,\mu_0 + N\,V(I(\nu_0)) + \frac{N(N-1)}{2}\bar{V}(\bar{I}(\bar{\nu}_0)) \quad \text{with} \\
\mu_0 &= \frac{m^2}{\mu_0} + \left[\frac{2Q^2(\nu_0 + N\bar{\nu}_0)}{\mu_0 N^2}\right]^{1/2} \\
I(\nu_0) &= \left[\frac{Q^2}{2N^2\mu_0(\nu_0 + N\bar{\nu}_0)}\right]^{1/4} \\
\bar{I}(\bar{\nu}_0) &= \left[\frac{2Q^2}{(N-1)^2\mu_0(\nu_0 + N\bar{\nu}_0)}\right]^{1/4}
\end{aligned}$$

N-body systems

Simplified formulation

With $X_0 = \sqrt{2\mu_0(\nu_0 + N\bar{\nu}_0)}$,

$$M(X_0) = N\sqrt{m^2 + \frac{Q}{N}X_0} + N\,V\left(\sqrt{\frac{Q}{NX_0}}\right) + \frac{N(N-1)}{2}\,\bar{V}\left(\sqrt{\frac{2Q}{(N-1)X_0}}\right)$$

where

$$X_0^2 = 2\sqrt{m^2 + \frac{Q}{N}X_0}\left[K\left(\sqrt{\frac{Q}{NX_0}}\right) + N\,\bar{K}\left(\sqrt{\frac{2Q}{(N-1)X_0}}\right)\right]$$

N-body systems

Upper and lower bounds

$g(x) = V(\sqrt{x})$ and $\bar{g}(x) = \bar{V}(\sqrt{x})$

- Nonrelativistic system ($m \to \infty$)
  If $g(x)$ and $\bar{g}(x)$ are concave, $M(X_0)$ is an upper bound
  If $g(x)$ and $\bar{g}(x)$ are convex, $M(X_0)$ is a lower bound

- Semirelativistic system ($m \ll M$)
  If $g(x)$ and $\bar{g}(x)$ are concave, $M(X_0)$ is an upper bound

N-body systems

Eigenfunctions

given in Jacobi coordinates as a product of $(N-1)$ oscillator states with a size depending on X_0

- Good parity
- Possible to fix angular momentum
- Possible to fix a symmetry

N-body systems

Eigenfunctions

given in Jacobi coordinates as a product of $(N-1)$ oscillator states with a size depending on X_0

- Good parity
- Possible to fix angular momentum
- Possible to fix a symmetry

Observables

- $\left\langle \sum_{i=1}^N \mathbf{p}_i^2 \right\rangle = Q\, X_0$
- $\left\langle \sum_{i=1}^N (\mathbf{r}_i - \mathbf{R})^2 \right\rangle = \dfrac{Q}{X_0}$
- $\left\langle \sum_{i<j=1}^N (\mathbf{r}_i - \mathbf{r}_j)^2 \right\rangle = \dfrac{N\, Q}{X_0}$

N-body systems

Duality relation $N \leftrightarrow 2$ bodies with $V(x) = 0$

$$M_{\text{UR}}^{(N)}(Q) = \frac{N(N-1)}{2} M_{\text{UR}}^{(2)}\left(\frac{1}{N-1}\sqrt{\frac{2}{N(N-1)}}Q\right)$$

N-body systems

Duality relation $N \leftrightarrow 2$ bodies with $V(x) = 0$

$$M_{\mathrm{UR}}^{(N)}(Q) = \frac{N(N-1)}{2} M_{\mathrm{UR}}^{(2)}\left(\frac{1}{N-1}\sqrt{\frac{2}{N(N-1)}}\,Q\right)$$

Duality relations $N \leftrightarrow 2$ bodies with $V(x) = 0$

$$E_{\mathrm{NR}}^{(N)}(m; Q) = \frac{N(N-1)}{2} E_{\mathrm{NR}}^{(2)}\left(\frac{N(N-1)^2}{2} m; Q\right)$$

- Exact for AFM approximations and $\bar{V}(r) \propto r^2$
- Errors around 5% for $\bar{V}(r) \propto r$ and $N = 3$

Application: baryons in large N_c limit

't Hooft limit (1974)

- Quarks are in $\square$ of SU(N_c) with $N_c \to \infty$
- $\alpha_S N_c \sim O(1)$
- Finite number of flavors
- Suppression of internal quark loops

Application: baryons in large N_c limit

't Hooft limit (1974)

- Quarks are in $\square$ of $SU(N_c)$ with $N_c \to \infty$
- $\alpha_S N_c \sim O(1)$
- Finite number of flavors
- Suppression of internal quark loops

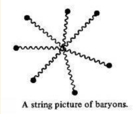
Baryon

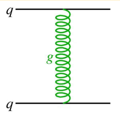
- N_c quarks in a totally antisymmetric color singlet
- Masses of heavy baryons $\sim O(N_c)$
- Same scaling expected for light baryons

Application: baryons in large N_c limit

Light baryon Hamiltonian (Witten, 1979)

$$
\begin{aligned}
m_q &= 0 \\
H_{\mathrm{B}} &= \sum_{i=1}^{N_c} \sqrt{\mathbf{p}_i^2} + \underbrace{\sigma \sum_{i=1}^{N_c} |\mathbf{r}_i - \mathbf{R}|} + \underbrace{\kappa \sum_{i<j=1}^{N_c} \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|}}
\end{aligned}
$$





$$
\begin{aligned}
\sigma &\sim O(1) \quad \text{fundamental string tension} \\
\kappa &= \frac{1}{2}\left(C_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} - 2C_{\square}\right)\frac{\alpha_0}{N_c} \quad \text{with} \quad \alpha_0 \sim O(1)
\end{aligned}
$$

Application: baryons in large N_c limit

Light baryon properties with $N_c \to \infty$

Band number: $K = \sum_{i=1}^{N_c-1}(2n_i + \ell_i) \sim O(1)$

$$M_{\mathrm{B}} \approx N_c \sqrt{\sigma\left(6 - \frac{\alpha_0}{\sqrt{2}}\right)} \sim O(N_c)$$

Regge trajectories: $M_{\mathrm{B}}^2 \approx \beta\, K + \gamma$ with N_c fixed

Contribution of $n_s (\ll N_c)$ strange quarks:

$$\Delta M_s \approx n_s \frac{m_s^2}{6\sqrt{\sigma}} \sqrt{6 - \frac{\alpha_0}{\sqrt{2}}} \sim O(1)$$

Mean square radius: $\langle r^2 \rangle \approx \frac{12-3\sqrt{2}\alpha_0}{8\sigma} \sim O(1)$

Publications of our team

- J. Phys. A: Math. Theor. **41** (2008) 275301
- J. Phys. A: Math. Theor. **41** (2008) 425301
- J. Phys. A: Math. Theor. **42** (2009) 245301
- J. Math. Phys. **50** (2009) 032102
- Phys. Rev. D **79** (2009) 094020
- Int. J. Mod. Phys. A **24** (2009) 4695
- J. Math. Phys. **51** (2010) 032104
- J. Phys. A: Math. Theor. **43** (2010) 265302
- Phys. Rev. D **82** (2010) 056008
- arXiv:1101.5222; arXiv:1102.1321