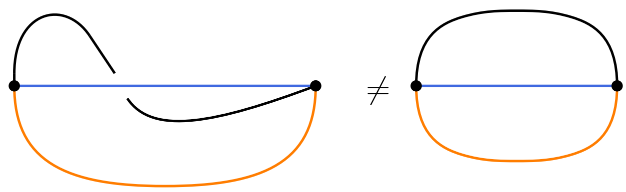
Session 2

Exercise 1. Bipartite maps, cut and join and KP hierarchy
The goal of the exercise is to prove that the partition function of bipartite maps – also alled Grothendieck dessins d'enfant – is a tau-function of the KP hierarchy, following a paper by Kazarian and Zograf (arXiv :1406.5976). We begin with some definitions.

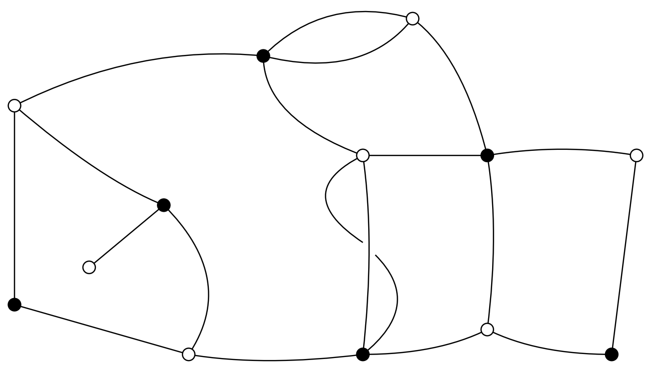
Definition 1. A *map* is a graph G where each vertex is endowed with a cyclic ordering of the incident half-edges.

For instance, the following graphs are the same, but they define two different maps. The notion of face is well-defined for a map. On the left hand side the map has one face ; on the right hand side, the map has 3 faces (we count the external face as well) :



We are interested in bipartite maps :

Definition 2. A *bipartite map* is a map with two kinds of vertices (we forbid isolated vertices) : black vertices $\bullet$ and white vertices $\circ$, such that each edge is adjacent to one black vertex and one white vertex.



We can orient the edges of a bipartite map from white vertices to black vertices, and say that an edge is adjacent to a face if the latter stands on the left of the edge with the given orientation. The *degree* of a face of a bipartite map is the number of edges adjacent to the face. It is also the number of white (resp. black) corners around the face.

We denote by $\mathfrak{B}(n, N_\bullet, N_\circ, \mathbf{f})$ the set of (non-necessarily connected) bipartite maps $\mathbf{m}$ with n edges, $N_\bullet$ (resp. $N_\circ$) black (resp. white) vertices, and f_i faces of degree i, and we enumerate those maps :

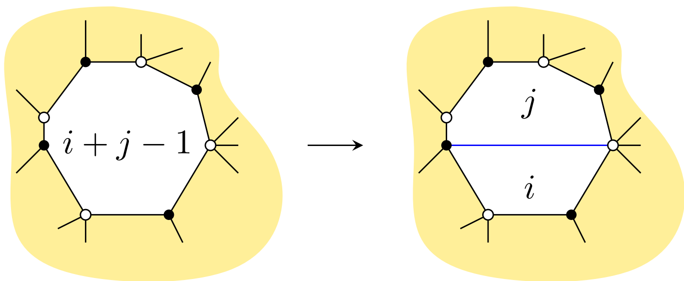
$$\mathcal{N}(n, N_\bullet, N_\circ, \mathbf{f}) \stackrel{\text{def}}{=} \sum_{\mathbf{m}\in\mathfrak{B}(n,N_\bullet,N_\circ,\mathbf{f})} \frac{1}{\#\mathrm{Aut}(\mathbf{m})}.$$

For instance, the bipartite map given above contributes to $\mathcal{N}(19, 6, 7, (1, 2, 1, 0, 1, 1, 0, 0, \dots))$ and has weight $u^{19}\, v_\circ^7 v_\bullet^6\, p_1 p_2^2 p_3 p_5 p_6$.
By convention, $\mathcal{N}(0, 0, 0, \mathbf{0}) = 1$ (it counts the empty map). We build the partition function

$$\tau(u, v_\bullet, v_\circ, \mathbf{p}) \stackrel{\text{def}}{=} \sum_{n,N_\bullet,N_\circ,\mathbf{f}} \mathcal{N}(n, N_\bullet, N_\circ, \mathbf{f})\, u^n\, v_\bullet^{N_\bullet}\, v_\circ^{N_\circ} \prod_i p_i^{f_i}. \qquad (1)$$

The idea of the exercise is to find an equation satisfied by τ by *removing an edge* from a bipartite map.

1. Consider the following procedure : for any $n \geqslant 0$ and for any bipartite map with $n + 1$ edges, choose one of the edges and consider that its weight is 1 (instead of u). Justify that enumerating the number of ways of doing so amounts to compute $\frac{\partial\tau}{\partial u}(u, v_\bullet, v_\circ, \mathbf{p})$.
2. We now look at the same procedure as in question 1, but in reverse direction : add a distinguished edge (of weight 1) to bipartites maps. There are several ways of doing so, and to see that, start with a bipartite map with n edges.
   (a) **First case**. We want to add the distinguished edge (in blue) inside a face of degree $i + j - 1$ in order to create two faces of degrees i and j, so that the degree i face stands on the left of the new edge :



   Let $\mathbf{m} \in \mathfrak{B}(n, N_\bullet, N_\circ, \mathbf{f})$; in how many ways can we add such an edge ? In the remaining of this question, we note γ this number.
   Deduce that, when running over the set $\mathfrak{B}(n, N_\bullet, N_\circ, \mathbf{f})$, the weighted number of maps with distinguished edge that we obtain is :

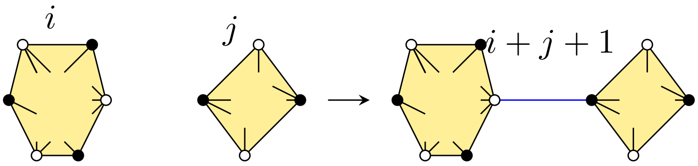
$$\gamma\mathcal{N}(n, N_\bullet, N_\circ, \mathbf{f}) u^n v_\bullet^{N_\bullet} v_\circ^{N_\circ} p_i^{f_i+1}\, p_j^{f_j+1}\, p_{i+j-1}^{f_{i+j-1}-1} \prod_{\ell\neq i,j,i+j-1} p_\ell^{f_\ell}.$$

   Show that, summing over $n, N_\bullet, N_\circ, \mathbf{f}$ and i, j, we get :

$$\sum_{i,j\geqslant 1} (i + j - 1) p_i\, p_j \frac{\partial}{\partial p_{i+j-1}} \tau(u, v_\bullet, v_\circ, \mathbf{p}).$$

   (b) **Second case**. We want to add the distinguished edge on a white vertex of a degree

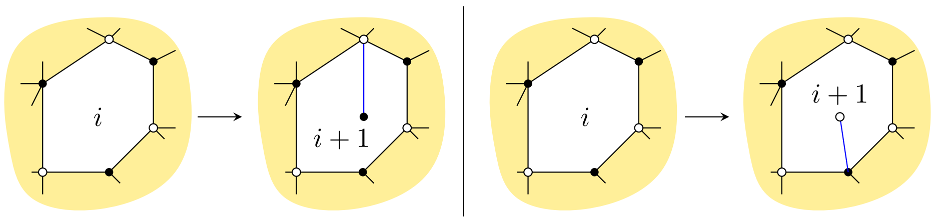
i face and a black vertex of a degree j face, to obtain a face of degree $i+j+1$:



Following the same kind of steps as in question 2.(a), show that enumerating this kind of edge adjunction amounts to compute

$$\sum_{i,j\geqslant 1} i\, j\, p_{i+j+1} \frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}\tau(u, v_\bullet, v_\circ, \mathbf{p}).$$

(c) **Third case**. The new edge is added to a white (resp. black) vertex of a face of degree i by creating also a new black (resp. white) vertex. The new face has degree $i+1$.



Find the operator to be applied to τ in this case.

(d) **Fourth case**. A new disconnected edge is added to the existing map. Find the operator to be applied to τ in this case.

3. Gathering questions 1 and 2, we obtain the *Cut-and-join* equation :

$$\frac{\partial \tau}{\partial u} = A \cdot \tau$$

where A is a sum of 4 explicit operators acting on τ. Give A.

4. Changing variables $t_j \stackrel{\text{def}}{=} \frac{p_j}{j}$, and using the boson-fermion correspondence, show that :

$$\tau = \langle 0|\mathrm{e}^{A(t)}\mathrm{e}^{u(v_\circ v_\bullet \alpha_{-1}+(v_\circ+v_\bullet)\Lambda_{-1}+M_{-1})}|0\rangle$$

5. Deduce that τ is a solution of the KP hierarchy.

Actually, we just proved that strictly monotone Hurwitz numbers satisfy the KP hierarchy!

Solution: 1. For a map with n edges, there are n ways of choosing a distinguished edge. Therefore, when $\mathbf{m}$ runs over $\mathfrak{B}(n, N_\bullet, N_\circ, \mathbf{f})$, we obtain $n\mathcal{N}(n, N_\bullet, N_\circ, \mathbf{f})$ maps with distinguished edge. We compute the partition function of such maps by taking care of putting a weight u^{n-1} for the n edges (the distinguished edge has weight 1) :

$$\sum_{n,N_\bullet,N_\circ,\mathbf{f}} n\mathcal{N}(n, N_\bullet, N_\circ, \mathbf{f})u^{n-1}v_\bullet^{N_\bullet}v_\circ^{N_\circ}\prod_i p_i^{f_i} = \frac{\partial}{\partial u}\sum_{n,N_\bullet,N_\circ,\mathbf{f}} \mathcal{N}(n, N_\bullet, N_\circ, \mathbf{f})u^{n}v_\bullet^{N_\bullet}v_\circ^{N_\circ}\prod_i p_i^{f_i}$$
$$= \frac{\partial \tau}{\partial u}$$

2.(a) For the first step, let $\mathbf{m} \in \mathfrak{B}(n, N_\bullet, N_\circ, \mathbf{f})$ and $i, j \geqslant 1$. There are f_{i+j-1} ways of choosing one face of degree $i+j-1$. Inside this face, there are $i+j-1$ ways of choosing a white corner. There is then a unique way of drawing an edge from this corner to a black corner of the same face so that the face on the left of the new edge has degree i.

The map that we obtain has $n+1$ edges (of total weight u^n since the new edge does not contribute) ; f_i+1 faces of degree i ; f_j+1 faces of degree j ; $f_{i+j-1}-1$ faces of degree $i+j-1$; and we do not change the other quantities. Therefore the weighted count becomes :

$$(i+j-1)f_{i+j-1}\mathcal{N}(n, N_\bullet, N_\circ, \mathbf{f})u^n v_\bullet^{N_\bullet} v_\circ^{N_\circ} p_i^{f_i+1} p_j^{f_j+1} p_{i+j-1}^{f_{i+j-1}-1} \prod_{\ell \neq i,j,i+j-1} p_\ell^{f_\ell}.$$

Summing over i, j and $n, N_\bullet, N_\circ, \mathbf{f}$, we get :

$$\begin{aligned}\sum_{i,j\geqslant 1} \sum_{n,N_\bullet,N_\circ,\mathbf{f}} &(i+j-1)f_{i+j-1}\mathcal{N}(n, N_\bullet, N_\circ, \mathbf{f})u^n v_\bullet^{N_\bullet} v_\circ^{N_\circ} p_i^{f_i+1} p_j^{f_j+1} p_{i+j-1}^{f_{i+j-1}-1} \prod_{\ell \neq i,j,i+j-1} p_\ell^{f_\ell} \\ &= \sum_{i,j\geqslant 1}(i+j-1)p_i p_j \frac{\partial}{\partial p_{i+j-1}} \sum_{n,N_\bullet,N_\circ,\mathbf{f}} \mathcal{N}(n, N_\bullet, N_\circ, \mathbf{f})u^n v_\bullet^{N_\bullet} v_\circ^{N_\circ} \prod_\ell p_\ell^{f_\ell} \\ &= \sum_{i,j\geqslant 1}(i+j-1)p_i p_j \frac{\partial}{\partial p_{i+j-1}}\tau.\end{aligned}$$

2.(b) Let us do the same steps as in previous question. Let $\mathbf{m} \in \mathfrak{B}(n, N_\bullet, N_\circ, \mathbf{f})$ and $i, j \geqslant 1$ (suppose for simplicity that $i \neq j$). There are $f_i \times f_j$ ways of choosing 2 faces of degree i and j respectively. In the face of degree i (resp. j), there are i ways of choosing a white (resp. black) corner. Once those choices are made, we can draw the new edge ! All in all, there are $f_i\, f_j\, i\, j$ choices for the distinguished new edge. Therefore, when running over $\mathfrak{B}(n, N_\bullet, N_\circ, \mathbf{f})$, we get $f_i\, f_j\, i\, j\, \mathcal{N}(n, N_\bullet, N_\circ, \mathbf{f})$ maps.
We now put everything in a generating series, and we should be cautious that, by adding the new edge, we replaced two faces of degree i, j by a face of degree $i+j+1$:

$$\sum_{n,N_\bullet,N_\circ,\mathbf{f}} f_i f_j i\, j \mathcal{N}(n, N_\bullet, N_\circ, \mathbf{f})u^n v_\bullet^{N_\bullet} v_\circ^{N_\circ} p_i^{f_i-1} p_j^{f_j-1} p_{i+j+1}^{f_{i+j+1}+1} \prod_{\ell \neq i,j,i+j+1} p_\ell^{f_\ell} = ijp_{i+j+1}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}\tau.$$

2.(c) Let $\mathbf{m} \in \mathfrak{B}(n, N_\bullet, N_\circ, \mathbf{f})$, assume that we attach the new edge to a black vertex in a face of degree i. There are if_i ways of choosing the black corner. In the new map, there are now f_i-1 faces of degree i, $f_{i+1}+1$ faces of degree $i+1$, and $N_\circ+1$ white vertices. The effect of such an operation on generating function is thus $v_\circ i\, p_{i+1}\frac{\partial \tau}{\partial p_i}$.
In the same manner if we attach the new edge to a white vertex $v_\bullet i\, p_{i+1}\frac{\partial \tau}{\partial p_i}$.
Summing over i, the operator acting on τ is then

$$(v_\circ + v_\bullet)\sum_{i\geqslant 1} ip_{i+1}\frac{\partial}{\partial p_i}\tau.$$

2.(d) For the last one, the operator is simply multiplication by $v_\circ v_\bullet p_1$.
3. We described the same operation in two ways : one way in question 1, that we put in the l.h.s., and another way in question 2, that we gather in the r.h.s. :

$$\frac{\partial \tau}{\partial u} = \underbrace{\left[v_\circ v_\bullet p_1 + (v_\circ + v_\bullet)\sum_{i\geqslant 1} ip_{i+1}\frac{\partial}{\partial p_i} + \sum_{i,j\geqslant 1}(i+j-1)p_i p_j\frac{\partial}{\partial p_{i+j-1}} + ijp_{i+j+1}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}\right]}_{=A}\tau.$$

4. Let us carry out the change of variable. The cut-and-join equation becomes :

$$\frac{\partial \tau}{\partial u} = \left[v_\circ v_\bullet t_1 + (v_\circ + v_\bullet) \sum_{i\geqslant 1} (i+1) t_{i+1} \frac{\partial}{\partial t_i} + \sum_{i,j\geqslant 1} ij t_i t_j \frac{\partial}{\partial t_{i+j-1}} + (i+j+1) t_{i+j+1} \frac{\partial}{\partial t_i}\frac{\partial}{\partial t_j} \right] \tau.$$

Let us denote by $|v\rangle \in \mathcal{F} = \Phi^{-1}(\tau)$ the preimage of τ under the boson-fermion isomorphism. From boson-fermion correspondence, we have :

$$\begin{aligned} \frac{\partial |v\rangle}{\partial u} &= \Bigg[v_\circ v_\bullet \alpha_{-1} + (v_\circ + v_\bullet) \underbrace{\sum_{i\geqslant 1} \alpha_{-i-1}\alpha_i}_{L_{-1}} + \underbrace{\sum_{i,j\geqslant 1} \alpha_{-i}\alpha_{-j}\alpha_{i+j-1} + \alpha_{-i-j-1}\alpha_i\alpha_j}_{M_{-1}} \Bigg] |v\rangle \\ &= \left[v_\circ v_\bullet \alpha_{-1} + (v_\circ + v_\bullet) L_{-1} + M_{-1} \right] |v\rangle \end{aligned}$$

Therefore :

$$|v\rangle = \mathrm{e}^{u[v_\circ v_\bullet \alpha_{-1} + (v_\circ + v_\bullet)L_{-1} + M_{-1}]} |v\rangle_{u=0}.$$

At $u = 0$, only the empty map contributes to τ, so $\tau(0, v_\bullet, v_\circ, \mathbf{p}) = 1 \Leftrightarrow |v\rangle_{u=0} = |0\rangle$.

$$\tau = \langle 0| \mathrm{e}^{A(\mathbf{t})} \mathrm{e}^{u[v_\circ v_\bullet \alpha_{-1} + (v_\circ + v_\bullet)L_{-1} + M_{-1}]} |0\rangle$$

5. The operators α_{-1}, L_{-1}, M_{-1} belong to $\widehat{gl(\infty)}$, so the partition function τ belongs to the orbit of the vacuum under the action of $\widehat{GL(\infty)}$: it satisfies the KP hierarchy !

Exercise 2. From Hirota to KP equation

Reminder : the Hirota equation for KP hierarchy can take this form

$$\operatorname*{Res}_{w=\infty} \mathrm{e}^{\sum_{j=1}^{\infty} w^j (t_j - s_j)} \tau(\mathbf{t} - [w^{-1}]) \tau(\mathbf{s} + [w^{-1}]) = 0, \tag{2}$$

where $[w^{-1}] = \left(w^{-1}, \frac{w^{-2}}{2}, \frac{w^{-3}}{3}, \dots \right)$.

1. Consider two functions f, g of infinitely many variables. Introduce the Hirota derivative as the following operator :

$$D_k\left(f \cdot g\right) \stackrel{\text{def}}{=} \frac{\partial}{\partial q_k} f(p_1, \dots, p_{k-1}, p_k + q_k, p_{k+1,\dots}) g(p_1, \dots, p_{k-1}, p_k - q_k, p_{k+1,\dots}) \Big|_{q_k=0}.$$

   By making the change of variables $t_i = p_i - q_i$, $s_i = p_i + q_i$, show that Hirota equation can be put in this form :

$$\operatorname*{Res}_{w=\infty} \left(\mathrm{e}^{-2\sum_{j=1}^{\infty} q_j w^j} \mathrm{e}^{-\sum_{j=1}^{\infty} \left(q_j + \frac{1}{j w^j} \right) D_j} \right) \tau \cdot \tau = 0.$$

2. Show that applying Hirota derivatives an odd number of times on $\tau \cdot \tau$ yields 0.
3. Consider that $q_k = q\, \delta_{k,1}$. We can view (2) as an infinite set of equations, one for each power of q.
   Prove that the coefficient of q^3 gives the following equation :

$$\left(D_1^4 + 3D_2^2 - 4D_1 D_3 \right) \tau \cdot \tau = 0. \tag{3}$$

4. Writing $\tau = \mathrm{e}^F$, rewrite (3) as an equation satisfied by F (the KP equation).

Solution: 1. (2) takes the following shape :

$$\operatorname*{Res}_{w=\infty} \mathrm{e}^{-2\sum_{j=1}^{\infty} q_j w^j} \tau(\mathbf{p}-\mathbf{q}-[w^{-1}])\tau(\mathbf{p}+\mathbf{q}+[w^{-1}]) = 0. \tag{4}$$

From the definition of Hirota derivative :

$$f(p_1,\ldots,p_{k-1},p_k+q_k,p_{k+1},\ldots)g(p_1,\ldots,p_{k-1},p_k+q_k,p_{k+1},\ldots) = \mathrm{e}^{q_k D_k} f(\mathbf{p})g(\mathbf{q}).$$

Therefore (4) can be written as

$$\operatorname*{Res}_{w=\infty} \mathrm{e}^{-2\sum_{j=1}^{\infty} q_j w^j} \mathrm{e}^{-\sum_{k=1}^{\infty}\left(q_k+\frac{w^{-k}}{k}\right)D_k} \tau\cdot\tau(\mathbf{p}) = 0. \tag{5}$$

2. We observe that $\tau(\mathbf{p}+\mathbf{q})\tau(\mathbf{p}-\mathbf{q})$ is even in $\mathbf{q}$, so applying Hirota derivatives an odd number of times to $\tau\cdot\tau$ gives zero.
3. We start from (5) with $q_k = q\delta_{k,1}$:

$$\operatorname*{Res}_{w=\infty} \mathrm{e}^{-2qw}\mathrm{e}^{-qD_1-\sum_{k=1}^{\infty}\frac{w^{-k}}{k}D_k}\tau\cdot\tau(\mathbf{p}) = 0.$$

The coefficient of q^3 in the integrand is a Laurent series in w^{-1}. Inside thies series, we are interested in the coefficient of w^{-1} since we take the residue. The terms of the form q^3w^{-1} in the integrand are :

$$\left[\frac{(-2qw)^3}{6}\Big(\frac{w^{-4}}{3}D_3D_1+\frac{w^{-4}}{4}D_2^2+\frac{w^{-4}}{24}D_1^4\Big)+\frac{(-2qw)^2}{2}\Big(\frac{qw^{-3}}{3}D_3D_1+\frac{qw^{-3}}{6}D_1^4\Big)\right.$$
$$\left.+(-2qw)\Big(\frac{q^2}{2}\frac{1}{2}D_1^4\Big)+\Big(\frac{q^3}{6}w^{-1}D_1^4\Big)\right]\tau\cdot\tau = \frac{q^3w^{-1}}{9}\Big(D_1^4+3D_2^2-4D_1D_3\Big)\tau\cdot\tau$$

(from question 2. we consider only the terms with an even number of Hirota derivatives). We obtain the desired equation.
4. We obtain the following equation

$$4\frac{\partial^2 F}{\partial p_3\partial p_1} - 3\frac{\partial^2 F}{\partial p_2^2} - 6\left(\frac{\partial^2 F}{\partial p_1^2}\right)^2 - \frac{\partial^4 F}{\partial p_1^4} = 0.$$

Hint :

$$Df\cdot g = f'g - fg', \qquad D^2 f\cdot g = f''g + fg'' - 2f'g', \quad \ldots$$