Session 1

Exercise 1. Wick's Theorem
For $i = 1, \dots, k$, set :

$$v_{n_i} = \sum_{n \geqslant 0} a_{i,n} \psi_{n_i - n}, \qquad w^*_{m_i} = \sum_{n \geqslant 0} b_{i,n} \psi^*_{m_i + n}.$$

1. Show that for all $i, j \in \{1, \dots, k\}$, $\langle 0 | v_{n_i} w^*_{m_j} | 0 \rangle$ is well-defined.
2. By induction, show Wick's theorem :

$$\langle 0 | v_{n_1} \dots v_{n_k} w^*_{m_k} \dots w^*_{m_1} | 0 \rangle = \det_{i,j=1,\dots,k} \langle 0 | v_{n_j} w^*_{m_i} | 0 \rangle. \tag{1}$$

Hint : justify that we can assume that $a_{i,n} = 0$ if $n_i - n > 0$ and $b_{j,n} = 0$ if $m_j + n > 0$; then anticommute $w^*_{m_k}$ to the left.

Solution: 1. Since $\langle 0 | \psi_i \psi^*_j | 0 \rangle = \delta_{i,j} \delta_{j<0}$, only a finite number of terms contribute to $\langle 0 | v_{n_i} w^*_{m_j} | 0 \rangle$:

$$\langle 0 | v_{n_i} w^*_{m_j} | 0 \rangle = \begin{cases} 0 \text{ if } m_j > 0, \\ \sum_{n=0}^{-m_j - \frac{1}{2}} a_{i, n_i - n} b_{j,n} \delta_{n_i - n, m_j + n} \text{ otherwise.} \end{cases}$$

2. First, if $i \in \mathbb{Z}_{\geqslant 0} + \frac{1}{2}$, $\langle 0 | \psi_i = 0$ and $\psi^*_i | 0 \rangle = 0$. Therefore, only the ψ_is and ψ^*_js with negative indices will contribute to the vacuum expectation value, so that we can assume that $a_{i,n} = 0$ if $n_i - n > 0$ and $b_{i,n} = 0$ if $m_i + n > 0$.

In those conditions, the following identity holds :

$$v_{n_i} w^*_{m_j} = \langle 0 | v_{n_i} w^*_{m_j} | 0 \rangle - w^*_{m_j} v_{n_j}. \tag{2}$$

Now, let us prove Wick's theorem. For $k = 1$, the identity is obviously true. Let us suppose that the identity holds up to order $k - 1$. We apply (2) successively to $v_{n_k} w^*_{m_k}$, $v_{n_{k-1}} w^*_{m_k}, \dots, v_{n_1} w^*_{m_k}$ in order to place $w^*_{m_k}$ on the left of the vacuum expectation value (where we have $\langle 0 | w^*_{m_k} = 0$). We get :

$$\begin{aligned} \langle 0 | v_{n_1} \dots v_{n_k} w^*_{m_k} \dots w^*_{m_1} | 0 \rangle &= \langle 0 | v_{n_k} w^*_{m_k} | 0 \rangle \langle 0 | v_{n_1} \dots v_{n_{k-1}} w^*_{m_{k-1}} \dots w^*_{m_1} | 0 \rangle \\ &\quad - \langle 0 | v_{n_1} \dots v_{n_{k-1}} w^*_{m_k} v_{n_k} w^*_{m_{k-1}} \dots w^*_{m_1} | 0 \rangle \\ &= \dots \\ &= \sum_{j=0}^{k-1} (-1)^j \langle 0 | v_{n_{k-j}} w^*_{m_k} | 0 \rangle \langle 0 | v_{n_1} \dots \widehat{v_{n_{k-j}}} \dots v_{n_k} w^*_{m_{k-1}} \dots w^*_{m_1} | 0 \rangle \end{aligned}$$

By induction hypothesis :

$$\langle 0 | v_{n_1} \dots \widehat{v_{n_{k-j}}} \dots v_{n_k} w^*_{m_{k-1}} \dots w^*_{m_1} | 0 \rangle = \det_{\substack{i,\ell=1,\dots,k \\ i \neq k, \ell \neq k-j}} \langle 0 | v_{n_\ell} w^*_{m_i} | 0 \rangle.$$

In the end, we recognise the expansion of the determinant with respect to a line, and we obtain Wick's theorem.

Exercise 2. We have defined $\alpha_n \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}+\frac{1}{2}} :\psi_{j-n}\psi_j:$ and $A(\mathbf{t}) \stackrel{\text{def}}{=} \sum_{n\geqslant 1} t_n\alpha_n$.

1. Show that, for all $k \in \mathbb{Z}+\frac{1}{2}$ and $n \in \mathbb{Z}_{>0}$:
$$[\alpha_n, \psi_k] = \psi_{k-n}, \qquad [\alpha_n, \psi_k^*] = -\psi_{k+n}^*.$$
2. Recall that the complete symmetric functions $h_n(\mathbf{t})$ are characterised by the equality $\mathrm{e}^{\sum_{i\geqslant 1} t_i z^i} = \sum_{n\geqslant 0} h_n(\mathbf{t})z^n$ in $\mathbb{C}[\mathbf{t}][[z]]$. Show that for $n>0$:
$$h_n(\mathbf{t}) = \sum_{m=1}^{n} \frac{1}{m!} \sum_{\substack{d_1,\ldots,d_m\geqslant 1\\ d_1+\cdots+d_m=n}} t_{d_1}\ldots t_{d_m}.$$
3. Using Hadamard identity
$$\mathrm{e}^{A(\mathbf{t})}\psi_k^{(*)}\mathrm{e}^{-A(\mathbf{t})} = \sum_{n\geqslant 0}\frac{1}{n!}\underbrace{\Big[A(\mathbf{t}),\big[A(\mathbf{t}),\ldots\big[A(\mathbf{t}),\psi_k^{(*)}\big]\ldots\big]\Big]}_{n\text{ commutations}}$$
and the previous questions, deduce :
$$\mathrm{e}^{A(\mathbf{t})}\psi_k\mathrm{e}^{-A(\mathbf{t})} = \sum_{n\geqslant 0} h_n(\mathbf{t})\psi_{k-n}, \qquad \mathrm{e}^{A(\mathbf{t})}\psi_k^*\mathrm{e}^{-A(\mathbf{t})} = \sum_{n\geqslant 0} h_n(-\mathbf{t})\psi_{k+n}^*. \tag{3}$$

Solution: 1. For $n>0$, $:\psi_{j-n}\psi_j^*: = \psi_{j-n}\psi_j^*$. We use $\{\psi_i\psi_j^*\} = \delta_{i,j}$ for $i,j\in\mathbb{Z}+\frac{1}{2}$:
$$[\alpha_n,\psi_k] = \sum_{j\in\mathbb{Z}+\frac{1}{2}}\left(\psi_{j-n}\psi_j^*\psi_k - \psi_k\psi_{j-n}\psi_j^*\right) = \sum_{j\in\mathbb{Z}+\frac{1}{2}}\Big(\delta_{k,j}\psi_{j-n}\underbrace{-\psi_{j-n}\psi_k\psi_j^* - \psi_k\psi_{j-n}\psi_j^*}_{\text{cancel out}}\Big)$$
$$= \psi_{k-n}.$$
Similarly for ψ_k^*.

2. We need to identify the coefficient of z^n in order to get h_n :
$$\mathrm{e}^{\sum_{i\geqslant 1} t_i z^i} = \sum_{m=0}^{\infty}\frac{1}{m!}\left(\sum_{i=1}^{\infty} t_i z^i\right)^m = \sum_{m=0}^{\infty}\frac{1}{m!}\sum_{d_1,\ldots,d_m\geqslant 1} t_{d_1}\ldots t_{d_m} z^{d_1+\cdots+d_m}$$
$$= \sum_{n\geqslant 0}\left(\sum_{m=0}^{n}\frac{1}{m!}\sum_{\substack{d_1,\ldots,d_m\geqslant 1\\ d_1+\cdots+d_m=n}} t_{d_1}\ldots t_{d_m}\right)z^n = \sum_{n\geqslant 0} h_n(\mathbf{t})z^n.$$

3. We show the result for ψ_k, the computations transpose easily to ψ_k^*. From question 1, we have $[A(\mathbf{t}),\psi_k] = \sum_{n\geqslant 1} t_n\psi_{k-n}$. Therefore :
$$\mathrm{e}^{A(\mathbf{t})}\psi_k\mathrm{e}^{-A(\mathbf{t})} = \sum_{m\geqslant 0}\frac{1}{m!}\underbrace{\Big[A(\mathbf{t}),\big[A(\mathbf{t}),\ldots\big[A(\mathbf{t}),\psi_k\big]\ldots\big]\Big]}_{m\text{ commutations}}$$
$$= \sum_{m\geqslant 0}\frac{1}{m!}\sum_{d_1,\ldots,d_m\geqslant 1} t_{d_1}\ldots t_{d_m}\psi_{k-d_1-\cdots-d_m}$$
$$= \sum_{n\geqslant 0}\underbrace{\left(\sum_{m=0}^{n}\frac{1}{m!}\sum_{\substack{d_1,\ldots,d_m\geqslant 1\\ d_1+\cdots+d_m=n}} t_{d_1}\ldots t_{d_m}\right)}_{=h_n(\mathbf{t})\text{ (question 2)}}\psi_{k-n}.$$

Exercise 3. Schur polynomials and the boson-fermion correspondence.
The isomorphism between the fermionic and the bosonic Fock spaces is :

$$\begin{aligned} \Phi: \ \mathcal{F} &\longrightarrow \mathcal{B}[z, z^{-1}] \\ |v\rangle &\longmapsto \sum_{\ell \in \mathbb{Z}} z^{\ell} \langle \ell | \mathrm{e}^{A(\mathbf{t})} | v \rangle . \end{aligned}$$

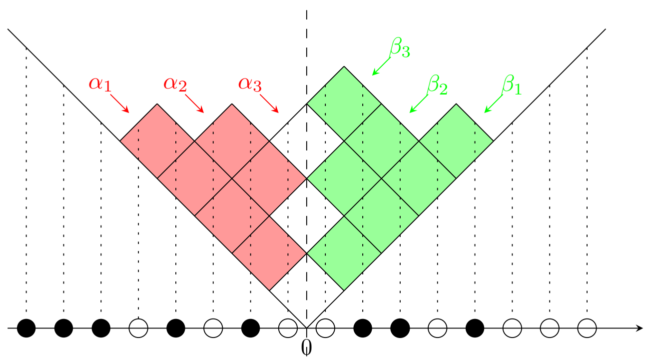
The goal of the exercise is to show that, for a partition λ, $\Phi(|0,\lambda\rangle)(\mathbf{t}) = s_\lambda(\mathbf{t})$, where the s_λ are the Schur polynomials.
Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$ be a partition and denote by d the number of diagonal boxes. We recall the Frobenius notation for the partition $\lambda = (\alpha_1, \ldots, \alpha_d | \beta_1, \ldots, \beta_d)$, where

- α_i is the number of boxes in the i^{th} column strictly under the diagonal ;
- β_j is the number of boxes in the j^{th} line striclty on the right of the diagonal.

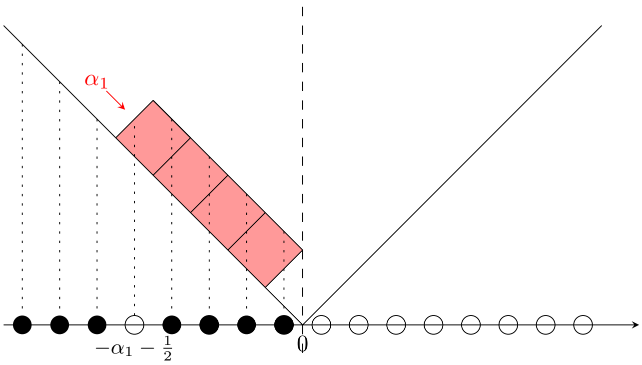
1. Show that
$$|0,\lambda\rangle = (-1)^{\alpha_1+\cdots+\alpha_d} \, \psi_{\beta_1+\frac{1}{2}} \cdots \psi_{\beta_d+\frac{1}{2}} \psi^*_{-\alpha_d-\frac{1}{2}} \cdots \psi^*_{-\alpha_1-\frac{1}{2}} |0\rangle .$$
2. Use Wick's theorem (1) and the conjugation formulas (3) to show that :
$$\Phi(|0,\lambda\rangle)(\mathbf{t}) = (-1)^{\alpha_1+\cdots+\alpha_d} \det_{i,j=1,\ldots,d} \sum_{\ell=0}^{\alpha_i} h_{\beta_j+\ell+1}(\mathbf{t}) h_{\alpha-\ell}(-\mathbf{t}).$$
3. Pieri's rule allows to express the the Schur polynomial of a hook diagram (a partition of the form $(\beta+1, 1^\alpha)$, or $(\alpha|\beta)$ in the Frobenius notation), while Giambelli's formula expresses the Schur polynomial of a partition in terms of Schur polynomials of hook diagrams :
$$\begin{aligned} s_{(\alpha|\beta)}(\mathbf{t}) &= (-1)^\alpha \sum_{\ell=0}^{\alpha} h_{\beta+\ell+1}(\mathbf{t}) h_{\alpha-\ell}(-\mathbf{t}) \quad \text{(Pieri's rule)}, \\ s_{(\alpha_1\ldots\alpha_d|\beta_1\ldots\beta_d)}(\mathbf{t}) &= \det_{i,j=1,\ldots,d} s_{(\alpha_i|\beta_j)}(\mathbf{t}) \quad \text{(Giambelli)}. \end{aligned}$$

   Use those identities to show that $\Phi(|0,\lambda\rangle)(\mathbf{t}) = s_\lambda(\mathbf{t})$.

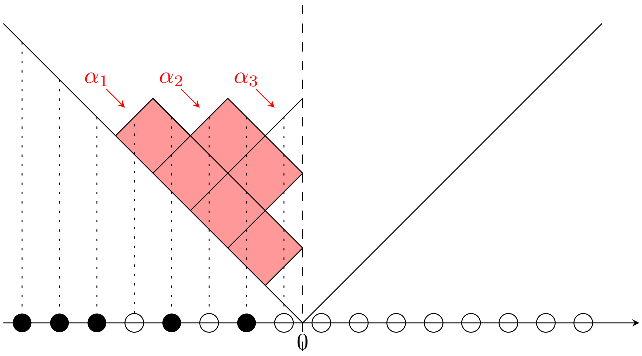
Solution: 1. Let $\lambda = (\alpha_1, \ldots, \alpha_d | \beta_1, \ldots, \beta_d)$. We want to produce such a state by applying fermionic operators on the vacuum :



We begin with the left part of the partition by acting with ψ_j^* on $|0\rangle$ in order to remove black stones from the Maya diagram. We first need to remove the black stone at position $-\alpha_1 - \frac{1}{2}$, by applying $(-1)^{\alpha_1}\psi^*_{-\alpha_1-\frac{1}{2}}$. This will produce the state :



Doing the same for the stones at positions $-\alpha_2 - \frac{1}{2}, \dots, -\alpha_d - \frac{1}{2}$, we get the state $(-1)^{\alpha_1+\dots+\alpha_d}\psi^*_{-\alpha_d-\frac{1}{2}} \dots \psi^*_{-\alpha_1-\frac{1}{2}}|0\rangle$:



Then, we add black stones on the right part of the Maya diagram by applying successively $\psi_{\beta_d+\frac{1}{2}}, \dots, \psi_{\beta_1+\frac{1}{2}}$ (there is no sign issue here), in order to get the desired Maya diagram.

2. From question 1, we have :

$$\begin{aligned}\Phi(|0,\lambda\rangle)(\mathbf{t}) &= (-1)^{\alpha_1+\dots+\alpha_d}\langle 0|\mathrm{e}^{A(\mathbf{t})}\psi_{\beta_1+\frac{1}{2}} \dots \psi_{\beta_d+\frac{1}{2}}\psi^*_{-\alpha_d-\frac{1}{2}} \dots \psi^*_{-\alpha_1-\frac{1}{2}}|0\rangle \\ &= (-1)^{\alpha_1+\dots+\alpha_d}\langle 0|\mathrm{e}^{A(\mathbf{t})}\psi_{\beta_1+\frac{1}{2}}e^{-A(\mathbf{t})} \dots e^{A(\mathbf{t})}\psi_{\beta_d+\frac{1}{2}}e^{-A(\mathbf{t})}e^{A(\mathbf{t})}\psi^*_{-\alpha_d-\frac{1}{2}}e^{-A(\mathbf{t})} \\ &\qquad \dots e^{A(\mathbf{t})}\psi^*_{-\alpha_1-\frac{1}{2}}e^{-A(\mathbf{t})}e^{A(\mathbf{t})}|0\rangle\end{aligned}$$

(we restrict to $\langle 0|$ since zero charge states are sent to z^0 polynomials under Φ).

For $i \in \{1, \dots, d\}$, we define $v_{\beta_i+\frac{1}{2}}$, $w^*_{-\alpha_i-\frac{1}{2}}$:

$$v_{\beta_i+\frac{1}{2}} = \mathrm{e}^{A(\mathbf{t})}\psi_{\beta_i+\frac{1}{2}}\mathrm{e}^{-A(\mathbf{t})}, \qquad w^*_{-\alpha_i-\frac{1}{2}} = \mathrm{e}^{A(\mathbf{t})}\psi^*_{-\alpha_i-\frac{1}{2}}\mathrm{e}^{-A(\mathbf{t})}.$$

Therefore we can recast the polynomial to compute as :

$$\Phi(|0,\lambda\rangle)(\mathbf{t}) = (-1)^{\alpha_1+\dots+\alpha_d}\langle 0|v_{\beta_1+\frac{1}{2}} \dots v_{\beta_d+\frac{1}{2}}w^*_{-\alpha_d-\frac{1}{2}} \dots w^*_{-\alpha_1-\frac{1}{2}}\underbrace{e^{A(\mathbf{t})}|0\rangle}_{=|0\rangle}$$

We now apply Wick's theorem (1) (we justify that vs and ws have the same shape as in Exercise 1 in a minute) :

$$\Phi(|0,\lambda\rangle)(\mathbf{t}) = (-1)^{\alpha_1+\dots+\alpha_d} \det_{i,j=1,\dots,d} \langle 0|v_{\beta_j+\frac{1}{2}} w^*_{-\alpha_i-\frac{1}{2}}|0\rangle.$$

From the conjugation relations (3), we get

$$v_{\beta_i+\frac{1}{2}} = \sum_{k\geqslant 0} h_k(\mathbf{t})\psi_{\beta_i+\frac{1}{2}-k}, \qquad w^*_{-\alpha_i-\frac{1}{2}} = \sum_{\ell\geqslant 0} h_\ell(-\mathbf{t})\psi^*_{-\alpha_i-\frac{1}{2}+\ell},$$

(this justifies that the vs and ws have the same shape as in Exercise 1). Therefore

$$\Phi(|0,\lambda\rangle)(\mathbf{t}) = (-1)^{\alpha_1+\dots+\alpha_d} \det_{i,j=1,\dots,d} \sum_{k,\ell\geqslant 0} h_k(\mathbf{t})h_\ell(-\mathbf{t})\langle 0|\psi_{\beta_j+\frac{1}{2}-k}\psi^*_{-\alpha_i-\frac{1}{2}+\ell}|0\rangle.$$

We have $\psi^*_j|0\rangle = 0$ if $j > 0$, so we can restrict the sum over ℓ between 0 and α. Using $\langle 0|\psi_i\psi^*_j|0\rangle = \delta_{i,j}\delta_{j<0}$ and the multilinearity of the determinant, we can recast the last expression into :

$$\Phi(|0,\lambda\rangle)(\mathbf{t}) = \det_{i,j=1,\dots,d} (-1)^{\alpha_i} \sum_{\ell=0}^{\alpha} h_{\beta_j+\ell+1}(\mathbf{t})h_{\alpha-\ell}(-\mathbf{t}).$$

3. By straightforward application of Pieri's rule and Giambelli's formula :

$$\begin{aligned}\Phi(|0,\lambda\rangle)(\mathbf{t}) &= \det_{i,j=1,\dots,d} (-1)^{\alpha_i} \sum_{\ell=0}^{\alpha} h_{\beta_j+\ell+1}(\mathbf{t})h_{\alpha-\ell}(-\mathbf{t}) \\ &\overset{\text{Pieri}}{=} \det_{i,j=1,\dots,d} s_{(\alpha_i|\beta_j)}(\mathbf{t}) \overset{\text{Giambelli}}{=} s_\lambda(\mathbf{t}).\end{aligned}$$

Exercise 4. Vertex operators
Reminder : we defined the vertex operators in $\mathcal{F}$ as

$$\psi(u) \overset{\text{def}}{=} \sum_{j\in\mathbb{Z}+\frac{1}{2}} \psi_j u^{-j-\frac{1}{2}}, \qquad \psi^*(u) \overset{\text{def}}{=} \sum_{j\in\mathbb{Z}+\frac{1}{2}} \psi^*_j u^{-j-\frac{1}{2}}.$$

We stated that, under the boson-fermion correspondence, the images of $\psi(u)$, $\psi^*(u)$ in $\mathcal{B}[z,z^{-1}]$ is

$$\Psi(u) \overset{\text{def}}{=} \mathrm{e}^{\xi(\mathbf{t},u^{-1})}\mathrm{e}^{-\xi(\widetilde{\partial},u)} z\, u^{-C-1}, \qquad \Psi^*(u) \overset{\text{def}}{=} \mathrm{e}^{-\xi(\mathbf{t},u)}\mathrm{e}^{\xi(\widetilde{\partial},u^{-1})} z^{-1}\, u^{-C-1},$$

where $\xi(\mathbf{t},u) \overset{\text{def}}{=} \sum_{k\geqslant 1} t_k u^k$, $\widetilde{\partial} \overset{\text{def}}{=} \left(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, \dots\right)$, and $u^{-C} : z^\ell f(\mathbf{t}) \to u^{-\ell} z^\ell f(\mathbf{t})$. The goal of this exercise is to prove the previous statement for $\Psi(u)$.

Wick's theorem for the vertex operators

$$\begin{aligned}\langle \ell|\psi(p_1)\dots\psi(p_n)\psi^*(q_n)\dots\psi^*(q_1)|\ell\rangle &= \det_{i,j=1,\dots,n} \frac{1}{1-p_iq_j}\frac{1}{p_i^\ell q_j^\ell} \\ &= \frac{1}{(p_1\dots p_n)^{\ell+1}(q_1\dots q_n)^\ell} \frac{\prod_{1\leqslant i<j\leqslant n}\left(\frac{1}{p_j}-\frac{1}{p_i}\right)(q_i-q_j)}{\prod_{1\leqslant i,j\leqslant n}\left(\frac{1}{p_i}-q_j\right)}.\end{aligned} \tag{4}$$

1. Show that
$$e^{A(\mathbf{t})}\psi(u)e^{-A(\mathbf{t})} = e^{\xi(\mathbf{t},u^{-1})}\psi(u), \tag{5}$$
$$\begin{aligned} e^{-\sum_{k\geqslant 1}\frac{u^k}{k}\alpha_k}\psi(p)e^{\sum_{k\geqslant 1}\frac{u^k}{k}\alpha_k} &= \Big(1-\frac{u}{p}\Big)\psi(p), \\ e^{-\sum_{k\geqslant 1}\frac{u^k}{k}\alpha_k}\psi^*(q)e^{\sum_{k\geqslant 1}\frac{u^k}{k}\alpha_k} &= \frac{1}{1-uq}\psi^*(q). \end{aligned} \tag{6}$$
Use (5) to show that :
$$\Phi(\psi(u)|v\rangle)(\mathbf{t}) = e^{\xi(\mathbf{t},u^{-1})}\sum_{\ell\in\mathbb{Z}} z^\ell\langle\ell|\psi(u)e^{A(\mathbf{t})}|v\rangle.$$
2. Prove the following :
$$\Psi(u)\Phi(|v\rangle)(\mathbf{t}) = e^{\xi(\mathbf{t},u^{-1})}\sum_{\ell\in\mathbb{Z}} z^\ell u^{-\ell}\langle\ell-1|e^{-\sum_{k\geqslant 1}\frac{u^k}{k}\alpha_k}e^{A(\mathbf{t})}|v\rangle.$$
It suffices then to show : $\langle\ell|\psi(u)|w\rangle = u^{-\ell}\langle\ell-1|e^{-\sum_{k\geqslant 1}\frac{u^k}{k}\alpha_k}|w\rangle\ \forall\ell\in\mathbb{Z}, |w\rangle\in\mathcal{F}$.
3. Suppose that $|w\rangle = \psi(p_2)\dots\psi(p_n)\psi^*(q_n)\dots\psi^*(q_1)|\ell\rangle$. Use (6) to show that :
$$u^{-\ell}\langle\ell-1|e^{-\sum_{k\geqslant 1}\frac{u^k}{k}\alpha_k}|w\rangle = u^{-\ell-1}\frac{\prod_{i=2}^n\left(\frac{1}{u}-\frac{1}{p_i}\right)}{\prod_{j=1}^n\left(\frac{1}{u}-q_j\right)}\operatorname*{Res}_{p_1=0} p_1^{\ell-1}\langle\ell|\psi(p_1)\dots\psi(p_n)\psi^*(q_n)\dots\psi^*(q_1)|\ell\rangle dp_1.$$
4. Use Wick's formula (4) to prove that for all $\ell\in\mathbb{Z}$:
$$\langle\ell|\psi(u)\psi(p_2)\dots\psi(p_n)\psi^*(q_n)\dots\psi^*(q_1)|\ell\rangle =$$
$$u^{-\ell-1}\frac{\prod_{i=2}^n\left(\frac{1}{u}-\frac{1}{p_i}\right)}{\prod_{j=1}^n\left(\frac{1}{u}-q_j\right)}\operatorname*{Res}_{p_1=0} p_1^{\ell-1}\langle\ell|\psi(p_1)\dots\psi(p_n)\psi^*(q_n)\dots\psi^*(q_1)|\ell\rangle dp_1.$$
Conclude.

Solution: 1. The computations are similar to the second question of exercise 2. For (5), we get :
$$\begin{aligned} e^{A(\mathbf{t})}\psi(u)e^{-A(\mathbf{t})} &= \sum_{k\in\mathbb{Z}+\frac{1}{2}} u^{-k-\frac{1}{2}}e^{A(\mathbf{t})}\psi_k e^{-A(\mathbf{t})} \\ &= \sum_{k\in\mathbb{Z}+\frac{1}{2}}\sum_{n\in\mathbb{Z}_{\geqslant 0}} u^{-k-\frac{1}{2}}h_n(\mathbf{t})\psi_{k-n} \\ &= \sum_{n\geqslant 0} h_n(\mathbf{t})u^{-n}\sum_{k\in\mathbb{Z}+\frac{1}{2}} u^{-k-\frac{1}{2}}\psi_k = e^{\xi(\mathbf{t},u^{-1})}\psi(u). \end{aligned}$$
For (6), let us do the computation for $\psi(p)$:
$$\begin{aligned} e^{-\sum_{k\geqslant 1}\frac{u^k}{k}\alpha_k}\psi(p)e^{\sum_{k\geqslant 1}\frac{u^k}{k}\alpha_k} &= \sum_{i\in\mathbb{Z}+\frac{1}{2}} p^{-i-\frac{1}{2}}\sum_{m\in\mathbb{Z}_{\geqslant 0}}\frac{1}{m!}\sum_{d_1,\dots,d_m\geqslant 1}\frac{-u^{d_1}}{d_1}\dots\frac{-u^{d_m}}{d_m}\psi_{i-d_1-\dots-d_m} \\ &= \sum_{i\in\mathbb{Z}+\frac{1}{2}} p^{-i-\frac{1}{2}}\sum_{n\in\mathbb{Z}_{\geqslant 0}} h_n\Big(-u,\frac{-u^2}{2},\frac{-u^3}{3},\dots\Big)\psi_{i-n} \\ &= e^{-\sum_{k\geqslant 1}\frac{1}{k}\frac{u^k}{p^k}}\sum_{i\in\mathbb{Z}+\frac{1}{2}} p^{-i-\frac{1}{2}}\psi_i = \Big(1-\frac{u}{p}\Big)\psi(p). \end{aligned}$$

Now :

$$\Phi(\psi(u)|v\rangle)(\mathbf{t}) = \sum_{\ell\in\mathbb{Z}} z^\ell \langle \ell|\mathrm{e}^{A(\mathbf{t})}\psi(u)|v\rangle \overset{(5)}{=} \sum_{\ell\in\mathbb{Z}} z^\ell \langle \ell|\mathrm{e}^{\xi(\mathbf{t},u^{-1})}\psi(u)\mathrm{e}^{A(\mathbf{t})}|v\rangle.$$

2. From the formula of $\Psi(u)$:

$$\begin{aligned}\Psi(u)\Phi(|v\rangle)(\mathbf{t}) =& \mathrm{e}^{\xi(\mathbf{t},u^{-1})}\sum_{\ell\in\mathbb{Z}} z^{\ell+1}u^{-\ell-1}\langle \ell|\mathrm{e}^{-\sum_{k\geqslant 1}\frac{u^k}{k}\frac{\partial}{\partial t_k}}\mathrm{e}^{\sum_{n\geqslant 1} t_n\alpha_n}|v\rangle \\ =& \mathrm{e}^{\xi(\mathbf{t},u^{-1})}\sum_{\ell\in\mathbb{Z}} z^{\ell}u^{-\ell}\langle \ell-1|\mathrm{e}^{-\sum_{k\geqslant 1}\frac{u^k}{k}\alpha_k}\mathrm{e}^{A(\mathbf{t})}|v\rangle\end{aligned}$$

where the second equality comes from $\frac{\partial}{\partial t_k}\mathrm{e}^{\sum_{n\geqslant 1} t_n\alpha_n} = \alpha_n$ and the fact that the α_ns commute with eachother for $n \geqslant 1$.

3. We use the conjugation relations (6) to commute $\mathrm{e}^{-\sum_{k\geqslant 1}\frac{u^k}{k}\alpha_k}$ on the right :

$$u^{-\ell}\langle \ell-1|\mathrm{e}^{-\sum_{k\geqslant 1}\frac{u^k}{k}\alpha_k}|w\rangle = u^{-\ell}\frac{\prod_{i=2}^{n}\left(1-\frac{u}{p_i}\right)}{\prod_{j=1}^{n}(1-uq_j)}\langle \ell-1|\psi(p_2)\dots\psi(p_n)\psi^*(q_n)\dots\psi^*(q_1)\underbrace{\mathrm{e}^{-\sum_{k\geqslant 1}\frac{u^k}{k}\alpha_k}|\ell\rangle}_{=|\ell\rangle}$$

We then transform the products to get the correct shape, notice also that $\langle \ell-1| = \langle \ell|\psi_{\ell-\frac{1}{2}}$, so we get :

$$u^{-\ell}\langle \ell-1|\mathrm{e}^{-\sum_{k\geqslant 1}\frac{u^k}{k}\alpha_k}|w\rangle = u^{-\ell-1}\frac{\prod_{i=2}^{n}\left(\frac{1}{u}-\frac{1}{p_i}\right)}{\prod_{j=1}^{n}\left(\frac{1}{u}-q_j\right)}\langle \ell|\psi_{\ell-\frac{1}{2}}\psi(p_2)\dots\psi(p_n)\psi^*(q_n)\dots\psi^*(q_1)|\ell\rangle.$$

Last, we remark that $\psi_{\ell-\frac{1}{2}}$ is the coefficient of $p_1^{-\ell}$ in $\psi(p_1)$, so we have

$$\psi_{\ell-\frac{1}{2}} = \operatorname*{Res}_{p_1=0} p_1^{\ell-1}\psi(p_1)\mathrm{d}p_1$$

and this yields the result.

4. This is a straight application of (4). We then obtain for all $ell \in \mathbb{Z}, |w\rangle \in \mathcal{F}$:

$$\langle \ell|\psi(u)|w\rangle = u^{-\ell}\langle \ell-1|\mathrm{e}^{-\sum_{k\geqslant 1}\frac{u^k}{k}\alpha_k}|w\rangle$$

and this proves that $\Phi(\psi(u)) = \Psi(u)$.