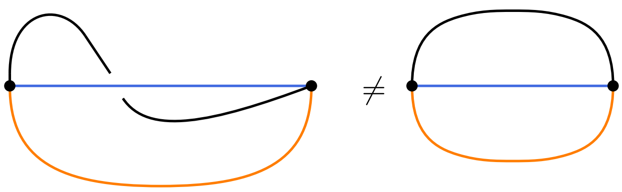
Session 2

Exercise 1. Bipartite maps, cut and join and KP hierarchy
The goal of the exercise is to prove that the partition function of bipartite maps – also alled Grothendieck dessins d'enfant – is a tau-function of the KP hierarchy, following a paper by Kazarian and Zograf (arXiv :1406.5976). We begin with some definitions.

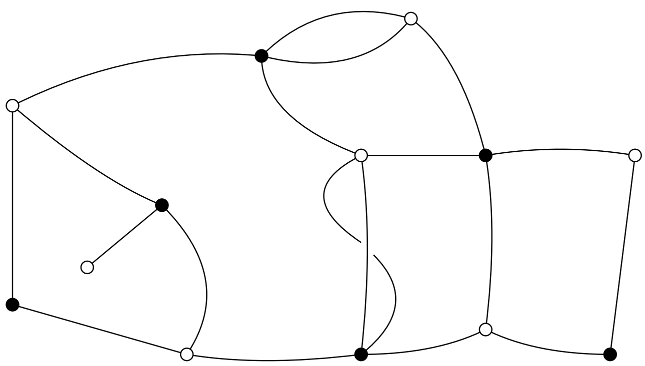
Definition 1. A *map* is a graph G where each vertex is endowed with a cyclic ordering of the incident half-edges.

For instance, the following graphs are the same, but they define two different maps. The notion of face is well-defined for a map. On the left hand side the map has one face ; on the right hand side, the map has 3 faces (we count the external face as well) :



We are interested in bipartite maps :

Definition 2. A *bipartite map* is a map with two kinds of vertices (we forbid isolated vertices) : black vertices • and white vertices ◦, such that each edge is adjacent to one black vertex and one white vertex.



We can orient the edges of a bipartite map from white vertices to black vertices, and say that an edge is adjacent to a face if the latter stands on the left of the edge with the given orientation. The *degree* of a face of a bipartite map is the number of edges adjacent to the face. It is also the number of white (resp. black) corners around the face.

We denote by $\mathfrak{B}(n, N_\bullet, N_\circ, \mathbf{f})$ the set of (non-necessarily connected) bipartite maps $\mathbf{m}$ with n edges, $N_\bullet$ (resp. $N_\circ$) black (resp. white) vertices, and f_i faces of degree i, and we enumerate those maps :

$$\mathcal{N}(n, N_\bullet, N_\circ, \mathbf{f}) \stackrel{\text{def}}{=} \sum_{\mathbf{m} \in \mathfrak{B}(n,N_\bullet,N_\circ,\mathbf{f})} \frac{1}{\#\mathrm{Aut}(\mathbf{m})}.$$

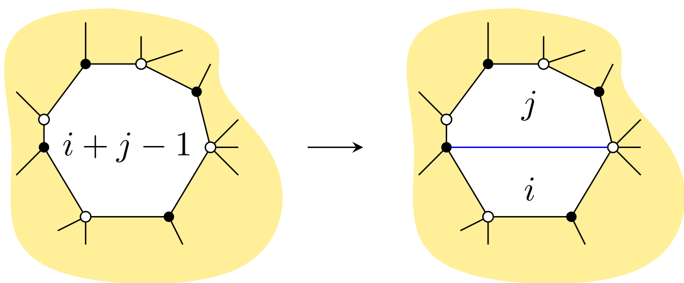
For instance, the bipartite map given above contributes to $\mathcal{N}(19, 6, 7, (1, 2, 1, 0, 1, 1, 0, 0, \dots))$ and has weight $u^{19}\, v_\circ^7 v_\bullet^6\, p_1 p_2^2 p_3 p_5 p_6$.

By convention, $\mathcal{N}(0, 0, 0, \mathbf{0}) = 1$ (it counts the empty map). We build the partition function

$$\tau(u, v_\bullet, v_\circ, \mathbf{p}) \stackrel{\text{def}}{=} \sum_{n,N_\bullet,N_\circ,\mathbf{f}} \mathcal{N}(n, N_\bullet, N_\circ, \mathbf{f})\, u^n\, v_\bullet^{N_\bullet}\, v_\circ^{N_\circ} \prod_i p_i^{f_i}. \tag{1}$$

The idea of the exercise is to find an equation satisfied by τ by *removing an edge* from a bipartite map.

1. Consider the following procedure : for any $n \geqslant 0$ and for any bipartite map with $n+1$ edges, choose one of the edge and consider that its weight is 1 (instead of u). Justify that enumerating the number of ways of doing so amounts to compute $\frac{\partial \tau}{\partial u}(u, v_\bullet, v_\circ, \mathbf{p})$.
2. We now look at the same procedure as in question 1, but in reverse direction : add a distinguished edge (of weight 1) to bipartites maps. There are several ways of doing so, and to see that, start with a bipartite map with n edges.
   (a) **First case**. We want to add the distinguished edge (in blue) inside a face of degree $i + j - 1$ in order to create two faces of degrees i and j, so that the degree i face stands on the left of the new edge :



Let $\mathbf{m} \in \mathfrak{B}(n, N_\bullet, N_\circ, \mathbf{f})$; in how many ways can we add such an edge ? In the remaining of this question, we note γ this number. there are $(i + j - 1)f_{i+j-1}$ ways of adding such an edge.

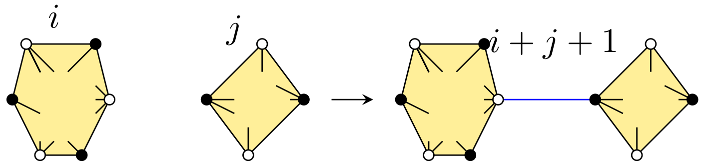
Deduce that, when running over the set $\mathfrak{B}(n, N_\bullet, N_\circ, \mathbf{f})$, the weighted number of maps with distinguished edge that we obtain is :

$$\gamma \mathcal{N}(n, N_\bullet, N_\circ, \mathbf{f}) u^n v_\bullet^{N_\bullet} v_\circ^{N_\circ} p_i^{f_i+1}\, p_j^{f_j+1}\, p_{i+j-1}^{f_{i+j-1}-1} \prod_{\ell \neq i,j,i+j-1} p_\ell^{f_\ell}.$$

Show that, summing over $n, N_\bullet, N_\circ, \mathbf{f}$ and i, j, we get :

$$\sum_{i,j\geqslant 1} (i + j - 1) p_i\, p_j \frac{\partial}{\partial p_{i+j-1}} \tau(u, v_\bullet, v_\circ, \mathbf{p}).$$

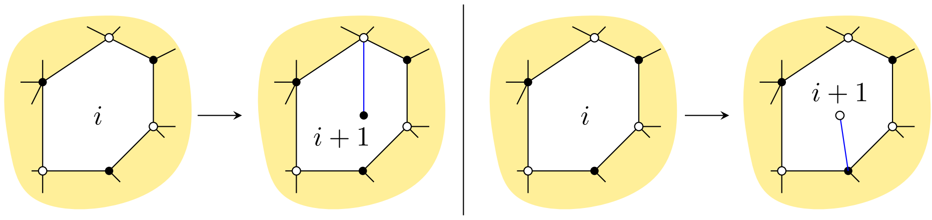
(b) **Second case**. We want to add the distinguished edge on a white vertex of a degree i face and a black vertex of a degree j face, to obtain a face of degree $i+j+1$:



Following the same kind of steps as in question 2.(a), show that enumerating this kind of edge adjunction amounts to compute

$$\sum_{i,j\geqslant 1} i\, j\, p_{i+j+1} \frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}\tau(u, v_\bullet, v_\circ, \mathbf{p}).$$

(c) **Third case**. The new edge is added to a white (resp. black) vertex of a face of degree i by creating also a new black (resp. white) vertex. The new face has degree $i+1$.



Find the operator to be applied to τ in this case.

(d) **Fourth case**. A new disconnected edge is added to the existing map.
Find the operator to be applied to τ in this case.

3. Gathering questions 1 and 2, we obtain the *Cut-and-join* equation :

$$\frac{\partial \tau}{\partial u} = A \cdot \tau$$

where A is a sum of 4 explicit operators acting on τ. Give A.

4. Changing variables $t_j \overset{\text{def}}{=} \frac{p_j}{j}$, and using the boson-fermion correspondence, show that :

$$\tau = \langle 0|\mathrm{e}^{A(t)}\mathrm{e}^{u(v_\circ v_\bullet \alpha_{-1} + (v_\circ + v_\bullet)\Lambda_{-1} + M_{-1})}|0\rangle$$

5. Deduce that τ is a solution of the KP hierarchy.

Actually, we just proved that strictly monotone Hurwitz numbers satisfy the KP hierarchy !

Exercise 2. From Hirota to KP equation
Reminder : the Hirota equation for KP hierarchy can take this form

$$\operatorname*{Res}_{w=\infty} \mathrm{e}^{\sum_{j=1}^{\infty} w^j (t_j - s_j)} \tau(\mathbf{t} - [w^{-1}])\tau(\mathbf{s} + [w^{-1}]) = 0, \tag{2}$$

where $[w^{-1}] = \left(w^{-1}, \frac{w^{-2}}{2}, \frac{w^{-3}}{3}, \dots\right)$.

1. Consider two functions f, g of infinitely many variables. Introduce the Hirota derivative as the following operator :

$$D_k\,(f \cdot g) \overset{\text{def}}{=} \frac{\partial}{\partial q_k} f(p_1, \dots, p_{k-1}, p_k + q_k, p_{k+1,\dots}) g(p_1, \dots, p_{k-1}, p_k - q_k, p_{k+1,\dots})\Big|_{q_k=0}.$$

By making the change of variables $t_i = p_i - q_i$, $s_i = p_i + q_i$, show that Hirota equation can be put in this form :

$$\operatorname*{Res}_{w=\infty} \left(\mathrm{e}^{-2\sum_{j=1}^{\infty} q_j w^j} \mathrm{e}^{-\sum_{j=1}^{\infty}\left(q_j + \frac{1}{j w^j}\right) D_j} \right) \tau \cdot \tau = 0.$$

2. Show that applying Hirota derivatives an odd number of times on $\tau \cdot \tau$ yields 0.
3. Consider that $q_k = q\,\delta_{k,1}$. We can view (2) as an infinite set of equations, one for each power of q.
   Prove that the coefficient of q^3 gives the following equation :

$$\left(D_1^4 + 3D_2^2 - 4D_1 D_3\right)\tau \cdot \tau = 0. \tag{3}$$

4. Writing $\tau = \mathrm{e}^F$, rewrite (3) as an equation satisfied by F (the KP equation).