Session 1

Exercise 1. Wick's Theorem
For $i = 1, \dots, k$, set :

$$v_{n_i} = \sum_{n \geqslant 0} a_{i,n} \psi_{n_i - n}, \qquad w^*_{m_i} = \sum_{n \geqslant 0} b_{i,n} \psi^*_{m_i + n}.$$

1. Show that for all $i, j \in \{1, \dots, k\}$, $\langle 0 | v_{n_i} w^*_{m_j} | 0 \rangle$ is well-defined.
2. By induction, show Wick's theorem :

$$\langle 0 | v_{n_1} \dots v_{n_k} w^*_{m_k} \dots w^*_{m_1} | 0 \rangle = \det_{i,j=1,\dots,k} \langle 0 | v_{n_j} w^*_{m_i} | 0 \rangle. \tag{1}$$

   Hint : justify that we can assume that $a_{i,n} = 0$ if $n_i - n > 0$ and $b_{j,n} = 0$ if $m_j + n > 0$; then anticommute $w^*_{m_k}$ to the left.

Exercise 2. We have defined $\alpha_n \overset{\text{def}}{=} \sum_{j \in \mathbb{Z} + \frac{1}{2}} : \psi_{j-n} \psi_j :$ and $A(\mathbf{t}) \overset{\text{def}}{=} \sum_{n \geqslant 1} t_n \alpha_n$.

1. Show that, for all $k \in \mathbb{Z} + \frac{1}{2}$ and $n \in \mathbb{Z}_{>0}$:

$$[\alpha_n, \psi_k] = \psi_{k-n}, \qquad [\alpha_n, \psi^*_k] = -\psi^*_{k+n}.$$

2. Recall that the elementary symmetric functions $h_n(\mathbf{t})$ are characterised by the equality $\mathrm{e}^{\sum_{i \geqslant 1} t_i z^i} = \sum_{n \geqslant 0} h_n(\mathbf{t}) z^n$ in $\mathbb{C}[\mathbf{t}][[z]]$. Show that for $n > 0$:

$$h_n(\mathbf{t}) = \sum_{m=1}^{n} \frac{1}{m!} \sum_{\substack{d_1, \dots, d_m \geqslant 1 \\ d_1 + \dots + d_m = n}} t_{d_1} \dots t_{d_m}.$$

3. Using Hadamard identity

$$\mathrm{e}^{A(\mathbf{t})} \psi^{(*)}_k \mathrm{e}^{-A(\mathbf{t})} = \sum_{n \geqslant 0} \frac{1}{n!} \underbrace{\big[A(\mathbf{t}), [A(\mathbf{t}), \dots [A(\mathbf{t}), \psi^{(*)}_k] \dots]\big]}_{n \text{ commutations}}$$

   and the previous questions, deduce :

$$\mathrm{e}^{A(\mathbf{t})} \psi_k \mathrm{e}^{-A(\mathbf{t})} = \sum_{n \geqslant 0} h_n(\mathbf{t}) \psi_{k-n}, \qquad \mathrm{e}^{A(\mathbf{t})} \psi^*_k \mathrm{e}^{-A(\mathbf{t})} = \sum_{n \geqslant 0} h_n(-\mathbf{t}) \psi^*_{k+n}. \tag{2}$$

Exercise 3. Schur polynomials and the boson-fermion correspondence.
The isomorphism between the fermionic and the bosonic Fock spaces is :

$$\begin{aligned} \Phi :\ \mathcal{F} &\longrightarrow \mathcal{B}[z, z^{-1}] \\ |v\rangle &\longrightarrow \sum_{\ell \in \mathbb{Z}} z^{\ell} \langle \ell | \mathrm{e}^{A(\mathbf{t})} | v \rangle. \end{aligned}$$

The goal of the exercise is to show that, for a partition λ, $\Phi(|0, \lambda\rangle)(\mathbf{t}) = s_\lambda(\mathbf{t})$, where the s_λ are the Schur polynomials.
Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)})$ be a partition and denote by d the number of diagonal boxes. We recall the Frobenius notation for the partition $\lambda = (\alpha_1, \dots, \alpha_d | \beta_1, \dots, \beta_d)$, where

- α_i is the number of boxes in the i^{th} column strictly under the diagonal ;
- β_j is the number of boxes in the j^{th} line striclty on the right of the diagonal.

1. Show that
$$|0,\lambda\rangle = (-1)^{\alpha_1+\cdots+\alpha_d}\,\psi_{\beta_1+\frac{1}{2}}\cdots\psi_{\beta_d+\frac{1}{2}}\psi^*_{-\alpha_d-\frac{1}{2}}\cdots\psi^*_{-\alpha_1-\frac{1}{2}}|0\rangle.$$
2. Use Wick's theorem (1) and the conjugation formulas (2) to show that :
$$\Phi\big(|0,\lambda\rangle\big)(\mathbf{t}) = (-1)^{\alpha_1+\cdots+\alpha_d}\det_{i,j=1,\ldots,d}\sum_{\ell=0}^{\alpha_i} h_{\beta_j+\ell+1}(\mathbf{t})h_{\alpha-\ell}(-\mathbf{t}).$$
3. Pieri's rule allows to express the the Schur polynomial of a hook diagram (a partition of the form $(\beta+1,1^\alpha)$, or $(\alpha|\beta)$ in the Frobenius notation), while Giambelli's formula expresses the Schur polynomial of a partition in terms of Schur polynomials of hook diagrams :
$$s_{(\alpha|\beta)}(\mathbf{t}) = (-1)^\alpha\sum_{\ell=0}^{\alpha} h_{\beta+\ell+1}(\mathbf{t})h_{\alpha-\ell}(-\mathbf{t}) \quad \text{(Pieri's rule)},$$
$$s_{(\alpha_1\ldots\alpha_d|\beta_1\ldots\beta_d)}(\mathbf{t}) = \det_{i,j=1,\ldots,d} s_{(\alpha_i|\beta_j)}(\mathbf{t}) \quad \text{(Giambelli)}.$$
Use those identities to show that $\Phi\big(|0,\lambda\rangle\big)(\mathbf{t}) = s_\lambda(\mathbf{t})$.

Exercise 4. (not corrected during the session) Vertex operators
Reminder : we defined the vertex operators in $\mathcal{F}$ as
$$\psi(u) \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}+\frac{1}{2}}\psi_j u^{-j-\frac{1}{2}}, \qquad \psi^*(u) \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}+\frac{1}{2}}\psi^*_j u^{-j-\frac{1}{2}}.$$
We stated that, under the boson-fermion correspondence, the images of $\psi(u)$, $\psi^*(u)$ in $\mathcal{B}[z,z^{-1}]$ is
$$\Psi(u) \stackrel{\text{def}}{=} \mathrm{e}^{\xi(\mathbf{t},u^{-1})}\mathrm{e}^{-\xi(\widetilde{\partial},u)}z\,u^{-C-1}, \qquad \Psi^*(u) \stackrel{\text{def}}{=} \mathrm{e}^{-\xi(\mathbf{t},u)}\mathrm{e}^{\xi(\widetilde{\partial},u^{-1})}z^{-1}\,u^{-C-1},$$
where $\xi(\mathbf{t},u) \stackrel{\text{def}}{=} \sum_{k\geqslant 1} t_k u^k$, $\widetilde{\partial} \stackrel{\text{def}}{=} \left(\frac{\partial}{\partial t_1},\frac{1}{2}\frac{\partial}{\partial t_2},\frac{1}{3}\frac{\partial}{\partial t_3},\ldots\right)$, and $u^{-C} : z^\ell f(\mathbf{t}) \to u^{-\ell}z^\ell f(\mathbf{t})$. The goal of this exercise is to prove the previous statement for $\Psi(u)$.

Wick's theorem for the vertex operators
$$\langle\ell|\psi(p_1)\ldots\psi(p_n)\psi^*(q_n)\ldots\psi^*(q_1)|\ell\rangle = \det_{i,j=1,\ldots,n}\frac{1}{1-p_iq_j}\frac{1}{p_i^\ell q_j^\ell}$$
$$= \frac{1}{(p_1\ldots p_n)^{\ell+1}(q_1\ldots q_n)^\ell}\frac{\prod_{1\leqslant i<j\leqslant n}\left(\frac{1}{p_j}-\frac{1}{p_i}\right)(q_i-q_j)}{\prod_{1\leqslant i,j\leqslant n}\left(\frac{1}{p_i}-q_j\right)}. \tag{3}$$

1. Show that
$$\mathrm{e}^{A(\mathbf{t})}\psi(u)\mathrm{e}^{-A(\mathbf{t})} = \mathrm{e}^{\xi(\mathbf{t},u^{-1})}\psi(u), \tag{4}$$
$$\begin{aligned}\mathrm{e}^{-\sum_{k\geqslant 1}\frac{u^k}{k}\alpha_k}\psi(p)\mathrm{e}^{\sum_{k\geqslant 1}\frac{u^k}{k}\alpha_k} &= \left(1-\frac{u}{p}\right)\psi(p),\\ \mathrm{e}^{-\sum_{k\geqslant 1}\frac{u^k}{k}\alpha_k}\psi^*(q)\mathrm{e}^{\sum_{k\geqslant 1}\frac{u^k}{k}\alpha_k} &= \frac{1}{1-uq}\psi^*(q).\end{aligned} \tag{5}$$
Use (4) to show that :
$$\Phi\big(\psi(u)|v\rangle\big)(\mathbf{t}) = \mathrm{e}^{\xi(\mathbf{t},u^{-1})}\sum_{\ell\in\mathbb{Z}} z^\ell\langle\ell|\psi(u)\mathrm{e}^{A(\mathbf{t})}|v\rangle.$$

2. Prove the following :

$$\Psi(u)\Phi(|v\rangle)(\mathbf{t}) = \mathrm{e}^{\xi(\mathbf{t},u^{-1})} \sum_{\ell \in \mathbb{Z}} z^{\ell} u^{-\ell} \langle \ell - 1 | \mathrm{e}^{-\sum_{k\geqslant 1} \frac{u^k}{k}\alpha_k} \mathrm{e}^{A(\mathbf{t})} |v\rangle.$$

It suffices then to show : $\langle \ell|\psi(u)|w\rangle = u^{-\ell}\langle \ell - 1|\mathrm{e}^{-\sum_{k\geqslant 1}\frac{u^k}{k}\alpha_k}|w\rangle\ \forall \ell \in \mathbb{Z}, |w\rangle \in \mathcal{F}$.

3. Suppose that $|w\rangle = \psi(p_2)\dot{\psi}(p_n)\psi^*(q_n)\ldots\psi^*(q_1)|\ell\rangle$. Use (5) to show that :

$$u^{-\ell}\langle \ell{-}1|\mathrm{e}^{-\sum_{k\geqslant 1}\frac{u^k}{k}\alpha_k}|w\rangle = u^{-\ell-1}\frac{\prod_{i=2}^{n}\left(\frac{1}{u}-\frac{1}{p_i}\right)}{\prod_{j=1}^{n}\left(\frac{1}{u}-q_j\right)}\operatorname*{Res}_{p_1=0} p_1^{\ell-1}\langle \ell|\psi(p_1)\ldots\psi(p_n)\psi^*(q_n)\ldots\psi^*(q_1)|\ell\rangle \mathrm{d}p_1.$$

4. Use Wick's formula (3) to prove that for all $\ell \in \mathbb{Z}$:

$$\langle \ell|\psi(u)\psi(p_2)\ldots\psi(p_n)\psi^*(q_n)\ldots\psi^*(q_1)|\ell\rangle =$$
$$u^{-\ell-1}\frac{\prod_{i=2}^{n}\left(\frac{1}{u}-\frac{1}{p_i}\right)}{\prod_{j=1}^{n}\left(\frac{1}{u}-q_j\right)}\operatorname*{Res}_{p_1=0} p_1^{\ell-1}\langle \ell|\psi(p_1)\ldots\psi(p_n)\psi^*(q_n)\ldots\psi^*(q_1)|\ell\rangle \mathrm{d}p_1.$$

Conclude.